Conformal Field Theory and Vertex Algebras

Julian F. Piribauer†

Supervisor: Prof. G. Felder

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†julianpi@student.ethz.ch

Abstract

This work intends to yield a brief introduction to conformal field theories, vertex algebras, and their mathematical preliminaries. The final objective is to construct a vertex algebra structure for representations of both the NEVEU-SCHWARZ and RAMOND algebra. Throughout the first four chapters, this work follows the topics published in *'A Mathematical Introduction to Conformal Field Theory'* (2008) by M. Schottenloher. We begin by classifying the conformal transformations of semi-RIEMANNian manifolds and by investigating the corresponding conformal groups. Special attention is payed to the world-sheet $\mathbb{S}^1 \times \mathbb{R}$ appearing in string theory, whose conformal group gives rise to the WITT algebra. In the second chapter, we investigate extensions of groups and LIE algebras, and furthermore explain their role in the process of quantisation. The findings of this chapter are used in the construction of the VIRASORO algebra, which – together with the study of its representations – forms the third chapter. The last two chapters cover the definition of a vertex algebra, a short introduction to supersymmetry in string theory, and the super-VIRASORO vertex algebra.

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CONTENTS

Chapter 1

Conformal Transformations

1.1 Classification of Conformal Transformations

Definition 1.1. Let (M, g) and (M', g') be two n-dimensional semi-RIEMANNian manifolds with $U \subset M$ and $U' \subset M'$ two open subsets of M respectively M'. We define a function $\phi : U \to U'$ to be a *conformal transformation* if ϕ is of maximal rank and if there exists a smooth function $\Omega : U \to \mathbb{R}^+$ with

$$\phi^* g' = \Omega^2 g.$$

Here, ϕ^* denotes the pullback of ϕ. The function Ω is often called *conformal factor*. The condition for ϕ being conformal can be rewritten with the use of local coordinates on both manifolds and $\left(\phi^* g'\right)_{\mu\nu}(a) \equiv g'_{ij}\left(\phi(a)\right)\partial_\mu\phi^i\,\partial_\nu\phi^j$ as

$$\Omega^2 g_{\mu\nu} = \left(g'_{ij} \circ \phi\right)\partial_\mu\phi^i\,\partial_\nu\phi^j. \tag{1.1}$$

In order to classify the conformal transformation belonging to the connected component of the conformal group that contains the identity transformation, it is useful to make use of the corresponding conformal KILLING factor.

Definition 1.2. A vector field X on $M \subset \mathbb{R}^{p,q}$ is called a *conformal vector field* if its corresponding local one-parameter group $\left(\phi^X_t\right)_{t\in\mathbb{R}}$ – defined via the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\phi^X(t,p)\right) = X\left(\phi^X(t,p)\right),$$

and initial condition $\phi^X_0 = \mathbb{1}$ – is conformal for all t in a neighbourhood of 0.

Theorem 1.1. *Let M be an open subset of $\mathbb{R}^{p,q}$ and $X = X^\nu\partial_\nu$ a conformal vector field on M. Then, there exists a smooth function $\kappa : M \to \mathbb{R}$ with*

$$X_{\mu,\nu} + X_{\nu,\mu} = \kappa\, g_{\mu\nu}. \tag{1.2}$$

Conversely, a function $\kappa : M \to \mathbb{R}$ satisfying this condition for some conformal KILLING field X is called a conformal KILLING factor.

Proof. Let $\left(\phi_t\right)_{t\in\mathbb{R}}$ be the local one-parameter group of X with Ω_t the corresponding conformal factor. Differentiating eq. (1.1) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Omega_t^2(a)\,g_{\mu\nu} &= g_{ij}\partial_\mu\dot\phi_0^i\,\partial_\nu\phi_0^j + g_{ij}\partial_\mu\phi_0^i\,\partial_\nu\dot\phi_0^j\\ &= g_{ij}\partial_\mu X^i(a)\,\delta_\nu^j + g_{ij}\partial_\nu X^j(a)\,\delta_\mu^i\\ &= \partial_\mu X_\nu(a) + \partial_\nu X_\mu(a).\end{aligned}$$

Thus, defining $\kappa(a) := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Omega_t^2(a)$ proves the theorem. □

Theorem 1.2. *A conformal* Killing *factor* $\kappa : M \to \mathbb{R}$ *fulfils*

$$(n-2)\,\kappa_{,\mu\nu} + g_{\mu\nu}\,\Delta\kappa = 0. \tag{1.3}$$

Proof. We assume that κ is a conformal Killing field, i.e. that there exists a conformal Killing field X with $X_{\mu,\nu} + X_{\nu,\mu} = g_{\mu\nu}\,\kappa$. It follows that

$$\begin{aligned}&g_{\mu\nu}\,\kappa_{,kl} - g_{k\nu}\,\kappa_{,l\mu} + g_{kl}\,\kappa_{,\mu\nu} - g_{\mu l}\,\kappa_{,\nu k}\\ &\qquad = \partial_k\partial_l\left(X_{\mu,\nu}+X_{\nu,\mu}\right) - \partial_l\partial_\mu\left(X_{k,\nu}+X_{\nu,k}\right) + \partial_\mu\partial_\nu\left(X_{k,l}+X_{l,k}\right) - \partial_\nu\partial_k\left(X_{\mu,l}+X_{l,\mu}\right)\\ &\qquad = 0,\end{aligned}$$

where we have used that partial derivatives commute on M. We contract the relation above with g^{kl} and find

$$\begin{aligned}0 &= g_{\mu\nu}\Delta\kappa - \delta_\nu^l\kappa_{,l\mu} + n\kappa_{,\mu\nu} - \delta_\mu^k\kappa_{,\nu k}\\ &= g_{\mu\nu}\Delta\kappa + (n-2)\kappa_{,\mu\nu}.\end{aligned}$$

□

1.1.1 Higher-dimensional semi-Riemannian manifolds

Let $n = p + q > 2$. Taking the trace of eq. (1.3) yields for $n \neq 1$

$$(n-2)\Delta\kappa + n\Delta\kappa = 0 \quad\Longrightarrow\quad \Delta\kappa = 0.$$

Since we are treating the case with $n > 2$, eq. (1.3) also tells us that $\kappa_{,\mu\nu} = 0$ for $\mu \neq \nu$. Thus, the derivatives of κ with respect to x^μ are constant and we conclude that every conformal Killing form κ must be of the form

$$\kappa(q) = \lambda + \alpha_\mu q^\mu, \qquad \lambda, \alpha_\mu \in \mathbb{R}.$$

Now that we have a general form of all conformal Killing forms, we can reconstruct the corresponding fields and their one-parameter groups. This will give us the complete set of conformal transformations in $\mathbb{R}^{p,q}$ with $n > 2$.

$\kappa = 0$ We immediately obtain $0 = g_{\mu\mu}\kappa = 2X_{\mu,\mu}$ for all conformal Killing fields X. Since partial derivatives commute on M and $\kappa_\sigma = 0$ we deduce from

$$\begin{aligned}X_{\mu,\nu\sigma} &= g_{\mu\nu}\kappa_{,\sigma} - X_{\nu,\mu\sigma}\\ &= X_{\sigma,\mu\nu}\\ &= -X_{\mu,\nu\sigma}\end{aligned}$$

that all second derivatives of X vanish. Thus, all fields X are of the form

$$X^\mu(q) = c^\mu + \omega^\mu_\nu q^\nu, \qquad c, \omega^\mu_\nu \in \mathbb{R}. \tag{1.4}$$

In the case where the coefficients ω^μ_ν vanish, we are left with the fields $X^\mu(q) = c^\mu$. The associated one-parameter groups are the *translations*

$$\phi^X_t(q) = q + tc.$$

If we insert the general solution of X (eq. (1.4)) into the defining relation of κ we find

$$0 = X_{\mu,\nu} + X_{\nu,\mu} = \omega_{\mu\nu} + \omega_{\nu\mu}.$$

Hence, $\omega \in \mathfrak{o}(p,q)$ and the one-parameter groups are the *orthogonal transformations* of the form

$$\phi^X_t(q) = \exp(t\omega)q, \qquad \exp(t\omega) \in \mathrm{O}(p,q).$$

κ = const. The discussion is analogue to the one above. Again, we find that X has to be an affine function of q with the only difference that for $\kappa = 2\lambda \in \mathbb{R}\backslash\{0\}$ we have $2\lambda = 2X^\mu_{,\mu}$ yielding $X^\mu(q) = \lambda q^\mu$. The corresponding one-parameter groups are the *dilatations*

$$\phi^X_t(q) = e^{t\lambda}q.$$

$\kappa = 4\langle q|b\rangle$ In the case where κ is a linear function of q we can show that the differential equation eq. (1.2) is solved by the fields

$$X^\mu(q) = 2\langle q|b\rangle q^\mu - \langle q|q\rangle b^\mu, \qquad b \in \mathbb{R}^n.$$

where the inner product on M is given by $\langle a|b\rangle = a_\mu b^\mu = g_{\mu\nu}a^\mu b^\nu$. Explicitly,

$$\begin{aligned} X^{\mu,\nu} + X^{\nu,\mu} &= 2b^\nu q^\mu + 2\langle q|b\rangle g^{\mu\nu} - 2q^\nu b^\mu + (\mu \leftrightarrow \nu) \\ &= 4\langle q|b\rangle g^{\mu\nu} \\ &= \kappa\, g^{\mu\nu}. \end{aligned}$$

There is no global one-parameter group associated to these kind of fields. Instead, we have to be satisfied with the local solutions of the form

$$\phi_t(q) = \frac{q - \langle q|q\rangle\, tb}{1 - 2\langle q|tb\rangle + \langle q|q\rangle\langle tb|tb\rangle}, \qquad t \in T_q \tag{1.5}$$

where $T_t \subset \mathbb{R}$ represents the maximal interval around 0 for which the denominator in eq. (1.5) does not vanish. To prove that $\phi_t(q)$ indeed solves the local flow equation for X we calculate

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi_t(q) &= -\langle q|q\rangle b + 2\langle q|b\rangle q \\ &= X(\phi_0(q)). \end{aligned}$$

These are the so-called *special conformal transformations*. We want to stress once more that $\phi := \phi_1$ does not possess a global continuation to $\mathbb{R}^n$ but at most to

$$M_1 := \{q \in \mathbb{R}^{p,q} \,|\, 1 - 2\langle q|b\rangle + \langle q|q\rangle\langle b|b\rangle \neq 0\}. \tag{1.6}$$

This fact will become important in section 1.2.

1.1.2 Euclidean Plane

We again start from eq. (1.3), which in two dimensions reduces to $\Delta\kappa = 0$. The defining equations for the conformal Killing fields are

$$\begin{aligned} \partial_0 X^0 &= \partial_1 X^1 = \frac{1}{2}\kappa \\ \partial_0 X^1 + \partial_1 X^0 &= 0. \end{aligned}$$

To simplify our discussion and connect the findings with something well-known, we associate M with the complex plane $\mathbb{C}$ by mapping $(x^0, x^1) \mapsto z = x^0 + \imath x^1$. The fields X split into the functions $X = (u, v) : M \to \mathbb{C}$ and simply describe the holomorphic transformations of M with

$$u_0 = v_1 = \frac{\kappa}{2}$$
$$u_1 + v_0 = 0.$$

$\kappa = 0$ For the case of a vanishing Killing factor, we find that the fields X are of the form

$$X(z) = c' + \imath\theta z, \qquad c' \in C, \theta \in \mathbb{R}.$$

The corresponding conformal transformations are the *Euclidean motions* $\phi_t(z) = ct + e^{\imath\theta t} z$ since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(z) &= c + \imath\theta e^{\imath\theta t} z \\ &= \left(c\, e^{-\imath\theta t}\partial_z + \imath\theta z\partial_z\right)\left(e^{\imath\theta t} z\right) \\ &= X\left(\phi_t(z)\right),\end{aligned}$$

where we identified $c = c'\, e^{\imath\theta t}$.

κ = **const.** Analogously to the case of vanishing conformal Killing factor, we find that $X(z) = \lambda z$ with the *dilatations*

$$\phi(z) = e^{\lambda} z$$

as their one-parameter group.

$\kappa = 4\,\mathrm{Re}(z\bar{b})$ The condition $\Delta\kappa = 0$ implies that the conformal Killing factor has to be of the form

$$\kappa = 4b_0\, x^0 + 4b_1\, x^1 = 4\,\mathrm{Re}(z\bar{b})$$

for some complex number b. The derivation of the corresponding conformal maps is identical to the case in $\mathbb{R}^{pq}$. We rewrite the general solution (cf. eq. (1.5))

$$\begin{aligned}\phi(z) &= \frac{z - |z|^2 b}{1 - 2\,\mathrm{Re}(z\bar{b}) + |z|^2|b|^2} \\ &= \frac{z(1-\bar{z}b)}{(1-z\bar{b})(1-\bar{z}b)} \\ &= \frac{z}{1-z\bar{b}}\end{aligned}$$

Conclusively, the connected component containing the identity of the group of conformal transformations in $\mathbb{R}^{2,0}$ is given by the Möbius transformations

$$\begin{aligned}\phi_S : \mathbb{C}\backslash\{cz+b=0\} &\longrightarrow \mathbb{C} \\ z &\mapsto \frac{az+b}{cz+d}, \qquad S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{C}).\end{aligned}$$

1.1.3 Minkowski Plane

Theorem 1.3. *Let $M \subset \mathbb{R}^{1,1}$ be a connected open subset. A smooth map $\phi = (u, v) : M \to \mathbb{R}^{1,1}$ is conformal if and only if $u_0^2 > v_0^2$ and*

$$\begin{matrix} u_0 = v_1 \\ u_1 = v_0 \end{matrix} \quad \text{or} \quad \begin{matrix} u_0 = -v_1 \\ u_1 = -v_0 \end{matrix} .$$

Proof. We start from the definition of a conformal transformation $\phi^* g = \Omega^2 g$ with $g = \mathrm{diag}(1, -1)$ which – using $\phi = (u, v)$ – reduces to

$$\begin{aligned} u_0^2 - v_0^2 &= \Omega^2 \\ u_0 u_1 - v_0 v_1 &= 0 \\ u_1^2 - v_1^2 &= -\Omega^2. \end{aligned} \tag{1.7}$$

With this we can readily prove the theorem:

"⇒" Let ϕ be such a conformal function. Then, eq. (1.7) implies

$$\begin{aligned} 0 &= \Omega^2 + 2u_0 u_1 - 2v_0 v_1 - \Omega^2 \\ &= u_0^2 - v_0^2 + 2u_0 u_1 - 2v_0 v_1 + u_1^2 - v_1^2 \\ &= (u_0 + u_1)^2 - (v_0 + v_1)^2 \end{aligned}$$

and we conclude $(u_0 + u_1) = \pm(v_0 + v_1)$. Furthermore,

$$\begin{aligned} 0 &= u_0^2 - u_0^2 + v_0 v_1 - u_0 u_1 \\ &= u_0^2 - u_0(u_0 + u_1) + v_0 v_1 \\ &= u_0^2 \mp u_0(v_0 + v_1) + v_0 v_1 \\ &= (u_0 \mp v_0)(u_0 \mp v_1). \end{aligned}$$

Thus ϕ either satisfies $(u_0 \mp v_1) = 0$ or $(u_0 \mp v_0) = 0$, where the latter contradicts the condition $u_0^2 - v_0^2 = \Omega^2 > 0$.

"⇐" If u and v satisfy the conditions from the theorem, we have

$$u_0 u_1 - v_0 v_1 = u_0 u_1 - (\pm u_1)(\pm u_0) = 0.$$

The statement follows with the definition of the conformal factor

$$\Omega^2 := u_0^2 - v_0^2 = -\left(u_1^2 - v_1^2\right) > 0.$$

□

Our final goal is to find an explicit form of the identity-component of the conformal transformations in $\mathbb{R}^{1,1}$. First, let us – for the sake of completeness – derive the corresponding Killing fields. In the Minkowski plane, the restriction $\Delta\kappa = 0$ (cf. eq. (1.3)) reduces to the one dimensional wave equation. The general solution is given by

$$\kappa(x^0, x^1) = f(x^+) + g(x^-),$$

where f and g are smooth functions and $x^\pm := x^0 \pm x^1$. Let $F(x^+) := \int^{x^+} \mathrm{d}t \frac{f(t)}{2}$ and $G(x^-) := \int^{x^-} \mathrm{d}t \frac{g(t)}{2}$. Then

$$X(x^0, x^1) = \begin{pmatrix} F(x^+) + G(x^-) \\ F(x^+) - G(x^-) \end{pmatrix}$$

satisfies $X_{\mu,\nu} + X_{\nu,\mu} = g_{\mu\nu}\kappa$.

We have shown that the identity-component of the conformal transformations in $\mathbb{R}^{1,1}$ consists of smooth mappings $\phi = (u, v)$ with $u_x = v_y$ and $u_y = v_x$. The solutions with opposite sign in theorem 1.3 represent orientation-inverting mappings. We can deduce the following explicit expression for the linear functions of this set.

Corollary 1.1. *The linear conformal mappings* $\phi : \mathbb{R}^{1,1} \to \mathbb{R}^{1,1}$ *inside the identity-component of the conformal maps in the* MINKOWSI *plane are of the form*

$$A_\phi = A_\pm(s,t) = e^t \begin{pmatrix} \pm\cosh s & \sinh s \\ \sinh s & \pm\cosh s \end{pmatrix}$$

with $s, t \in \mathbb{R}$.

Proof. For a general linear map $\phi = (u, v)$ in $\mathbb{R}^2$ we make the Ansatz

$$A_\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

where $u = ax^0 + bx^1$ and $v = cx^0 + dx^1$. With theorem 1.3 we know that

$$a^2 = u_0^2 > v_0^2 = c^2$$
$$a = u_0 = v_1 = d$$
$$b = u_1 = v_0 = c.$$

Since $\exp : \mathbb{R} \to \mathbb{R}^+$ and $\sinh : \mathbb{R} \to \mathbb{R}$ are bijective functions, there exists exactly one $t \in \mathbb{R}$ and one $s \in \mathbb{R}$, such that $\exp(2t) = a^2 - c^2$ and $\sinh s = \exp(-t)c$ respectively $c = \sinh s \exp t$. It follows that $a^2 = \exp(2t)(1 + \sinh^2 s) = \exp(2t)\cosh^2 s$ or $a = \pm \exp t \cosh s$. □

1.2 Conformal Compactification of $\mathbb{R}^{p,q}$

In chapter 1 we characterised the conformal transformations $\phi : M \to R^{p,q}$ for different values of p and q. Among those transformations there were some that could not be continued to global transformations (cf. eq. (1.6)). In this section we will develop the *conformal compactification* $N^{p,q}$ of $\mathbb{R}^{p,q}$ with which we will be able to assign to each conformal transformation $\phi : M \to \mathbb{R}^{p,q}$ a conformal diffeomorphism $\hat{\phi} : N^{p,q} \to N^{p,q}$. In $\mathbb{R}^{p,q}$ with $n = p + q > 2$, this procedure is fairly natural, whereas both two-dimensional cases need some special consideration. The conformal compactification of the MINKOWSKI plane will be dealt with in detail in section 1.3.

Definition 1.3. We denote the connected component containing the identity of the group of conformal diffeomorphisms $\hat{\phi} : N^{p,q} \to N^{p,q}$ between the conformal compactification $N^{p,q}$ of $\mathbb{R}^{p,q}$ as the *conformal group* $\mathrm{Conf}(\mathbb{R}^{p,q})$.

In the following we will use the notation $x^2 = g_{ab}x^a x^b$ for any vector $x \in \mathbb{R}^{m,n}$.

Definition 1.4. The $(n+1)$-dimensional real *projective space* is given by

$$\mathbb{P}_{n+1}(\mathbb{R}) := \left(\mathbb{R}^{n+2}\setminus\{0\}\right)/\sim$$
$$x \sim y \iff \exists \lambda \in \mathbb{R} : x = \lambda y.$$

We denote the quotient map corresponding to the equivalence relation above as $\gamma : \mathbb{R}^{n+2}\backslash\{0\} \to \mathbb{P}_{n+1}(\mathbb{R})$ with $\gamma\left(x^0,\ldots,x^n\right) = \left(x^0 : \ldots : x^n\right)$. Here we introduced the *homogeneous coordinates* indicated by the use of colons. The space $\mathbb{R}^{p,q}$ can be embedded into the projective space $\mathbb{P}_{n+1}(\mathbb{R})$ via

$$\begin{aligned} \imath : \mathbb{R}^{p,q} &\longrightarrow \mathbb{P}_{n+1}(\mathbb{R}) \\ x &\longmapsto \left(\frac{1-x^2}{2} : x^1 : \ldots : x^n : \frac{1+x^2}{2}\right). \end{aligned}$$

Corollary 1.2. *The closure of the embedding of* $\mathbb{R}^{p,q}$ *in the projective space* $\mathbb{P}_{n+1}(\mathbb{R})$ *is the quadric*

$$\overline{\imath(\mathbb{R}^{p,q})} = N^{p,q} := \left\{\xi \in \mathbb{P}_{n+1}(\mathbb{R}) \,|\, \xi^2 = 0\right\}.$$

Proof.

"$\subset$" With the definition of the projective embedding $\imath$ in eq. (1.8) we have for $x \in \mathbb{R}^{p,q}$

$$\begin{aligned} \imath(x)^2 &= \left(\frac{1-x^2}{2}\right)^2 + \underbrace{(x^1)^2 + \ldots - (x^n)^2}_{=x^2} - \left(\frac{1+x^2}{2}\right)^2 \\ &= 0. \end{aligned}$$

Thus $\imath(x) \in N^{p,q}$.

"$\supset$" Let $\xi = \left(\xi^0 : \ldots : \xi^{n+1}\right) \in N^{p,q}\backslash\imath(\mathbb{R}^{p,q})$. For $\lambda := \xi^0 + \xi^{n+1} \neq 0$, we have

$$\begin{aligned} \imath\left(\frac{1}{\lambda}\left(\xi^1,\ldots,\xi^n\right)\right) &= \left(\frac{1-\frac{\tilde{\xi}^2}{\lambda^2}}{2} : \frac{\xi^1}{\lambda} : \ldots : \frac{\xi^n}{\lambda} : \frac{1+\frac{\tilde{\xi}^2}{\lambda^2}}{2}\right) \\ &= \left(\frac{\left(\xi^0+\xi^{n+1}\right)^2 - \tilde{\xi}^2}{2\left(\xi^0+\xi^{n+1}\right)} : \xi^1 : \ldots : \xi^n : \frac{\left(\xi^0+\xi^{n+1}\right)^2 + \tilde{\xi}^2}{2\left(\xi^0+\xi^{n+1}\right)}\right) \\ &= \left(\frac{2\xi^0\xi^{n+1} - \xi^2 + 2(\xi^0)^2}{2\left(\xi^0+\xi^{n+1}\right)} : \xi^1 : \ldots : \xi^n : \frac{2\xi^0\xi^{n+1} + \xi^2 + 2(\xi^{n+1})^2}{2\left(\xi^0+\xi^{n+1}\right)}\right) \\ &= \left(\xi^0 : \ldots : \xi^{n+1}\right) \qquad \in \imath(\mathbb{R}^{p,q}), \end{aligned}$$

where we defined $\tilde{\xi} := \left(\xi^1 : \ldots : \xi^n\right)$ and used $\xi^2 = 0$ since $\xi \in N^{p,q}$. We conclude that $\xi^0 + \xi^{n+1} = 0$. Let now $\xi \in N^{p,q}$. There exist positive sequences $(\delta_k)_{k\in\mathbb{N}}$ and $(\epsilon_k)_{k\in\mathbb{N}}$ converging to zero with $2\xi^1\epsilon_k + \epsilon_k^2 = 2\xi^{n+1}\delta_k + \delta_k^2$ for all $k \in \mathbb{N}$, such that

$$\xi_k := \left(\xi^0 : \xi^1 + \epsilon_k : \xi^2 : \ldots : \xi^n : \xi^{n+1} + \delta_k\right) \qquad , n \in \mathbb{N}$$

still satisfies $(\xi_k)^2 = 0$, i.e. so that the sequence is contained in $N^{p,q}$. Since the first and last entry add up to a nonzero number, we have $\xi_k \in \imath(\mathbb{R}^{p,q})$ and with ξ_k approaching ξ, we conclude that $\xi \in \overline{\imath(\mathbb{R}^{p,q})}$.

□

Corollary 1.3. *Restricting the quotient map* $\gamma : \mathbb{R}^{n+2} \to \mathbb{P}_{n+1}(\mathbb{R})$ *to* $\mathbb{S}^p \times \mathbb{S}^q$ *yields a smooth 2-to-1 covering*

$$\pi := \gamma\Big|_{\mathbb{S}^p\times\mathbb{S}^q} : \mathbb{S}^p \times \mathbb{S}^q \longrightarrow N^{p,q}.$$

Proof. Since $\mathbb{S}^p \times \mathbb{S}^q \subset \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} = \mathbb{R}^{n+2}$ the map π is well-defined. If there are two elements $\xi, \xi' \in \mathbb{S}^p \times \mathbb{S}^q$ with $\gamma(\xi) = \gamma(\xi')$ then there exists a $\lambda \in \mathbb{R}\backslash\{0\}$ with $\xi' = \lambda\xi$. In fact, this is only possible for $\lambda = \pm 1$. For any $P = (\xi^0 : \ldots : \xi^{n+1}) \in N^{p,q}$ we have

$$0 = P^2 = \sum_{j=0}^{p} (\xi^j)^2 - \sum_{j=p+1}^{n+1} (\xi^j)^2.$$

We define $r := \sqrt{\sum_{j=0}^{p}(\xi^j)^2}$ and find that

$$\gamma^{-1}(\xi) = \left\{ \pm\frac{1}{r}\left(\xi^0,\dots,\xi^{n+1}\right)\right\} \in \mathbb{S}^p \times \mathbb{S}^q \qquad \forall \xi \in N^{p,q}.$$

□

With the local diffeomorphism $\pi : \mathbb{S}^p \times \mathbb{S}^q \to N^{p,q}$ we can adapt the metric of $\mathbb{R}^{p,q}$ on $N^{p,q}$.

Definition 1.5. The quadric $N^{p,q}$ with the semi-RIEMANNian metric induced by the embedding π is called the *conformal compactification* of $\mathbb{R}^{p,q}$.

Theorem 1.4. *For every element* $\Lambda \in \mathrm{O}(p+1, q+1)$ *we define the map*

$$\begin{aligned} \Psi = \Psi_\Lambda : N^{p,q} &\to N^{p,q}, \\ \xi &\longmapsto \gamma(\Lambda\xi), \end{aligned}$$

which is conformal and a diffeomorphism of $N^{p,q}$*. Its inverse exists for all* Λ *and is given by* $(\Psi_\Lambda)^{-1} = \Psi_{\Lambda^{-1}}$*. The relation* $\Psi_\Lambda = \Psi_{\Lambda'}$ *implies* $\Lambda' = \pm\Lambda$.

Proof. Let us first verify that Ψ_Λ is well-defined for every $\Lambda \in \mathrm{O}(p+1, q+1)$. First, we check that the image of Ψ_Λ is contained in $N^{p,q}$. For this to be true we require $\gamma(\Lambda\xi)^2 = 0$ or equivalently $(\Lambda\xi)^2 = 0$:

$$(\Lambda\xi)^2 = g(\Lambda\xi, \Lambda\xi) = g(\xi,\xi) = 0 \qquad \forall \xi \in N^{p,q}.$$

Second, the mapping must not depend on the choice of the representative ξ. This is fulfilled naturally by the definition of γ. Since we constructed the conformal compactification $N^{p,q}$ by adopting the metric of $\mathbb{R}^{p+1,q+1}$ – which is invariant under the transformations $\Lambda \in \mathrm{O}(p+1, q+1)$ – the mappings Ψ_Λ are indeed conformal.

Of course, $(\Psi_\Lambda)^{-1} = \Psi_{\Lambda^{-1}}$. The relation $\Psi_\Lambda = \Psi_{\Lambda'}$ reduced to $\gamma(\Lambda\xi) = \gamma(\Lambda'\xi)$ for all $\xi \in \mathbb{R}^{p,q}$ with $\xi^2 = 0$. Thus $\exists r \in \mathbb{R}\backslash\{0\} : \Lambda' = r\Lambda$. Due to the orthogonality of both Λ and Λ', this condition is restricted further to $r = \pm 1$. □

Before we start investigating the structure of the conformal group of $\mathbb{R}^{p,q}$, we end this general discussion of conformal compactification with the following definition.

Definition 1.6. We call $\hat{\phi} : N^{p,q} \to N^{p,q}$ the *conformal continuation* of the conformal transformation $\phi : M \to \mathbb{R}^{p,q}$ on the open subset $M \subset \mathbb{R}^{p,q}$, if it is a conformal diffeomorphism and if

$$\iota(\phi(x)) = \hat{\phi}(\iota(x)). \tag{1.9}$$

Visually, $\hat{\phi}$ is a conformal continuation of ϕ if the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\ \phi\ } & \mathbb{R}^{p,q} \\ \iota\downarrow & & \downarrow\iota \\ N^{p,q} & \xrightarrow[\ \hat{\phi}\ ]{} & N^{p,q}. \end{array}$$

We are now ready to prove the following theorem.

Theorem 1.5. *Every conformal transformation* $\phi : M \to \mathbb{R}^{p,q}$ *on an open subset* $M \subset \mathbb{R}^{p,q}$ *has a unique conformal continuation* $\hat{\phi} : N^{p,q} \to N^{p,q}$. *The group of all these conformal continuations is isomorphic to* $\mathrm{O}(p+1, q+1)\backslash\{\pm 1\}$. *Thus, the conformal group – the identity-component of this group – is per definition isomorphic to* $\mathrm{SO}(p+1, q+1)$.

Proof. For all the conformal transformations $\phi : M \to \mathbb{R}^{p,q}$ found in section 1.1.1, we need to find corresponding conformal continuations $\hat{\phi} : N^{p,q} \to N^{p,q}$ that satisfy eq. (1.9).

Orthogonal transformations. Let $\phi(x) = \Lambda' x$ with $\Lambda' \in \mathrm{O}(p, q)$. We define

$$\Lambda_{\Lambda'} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \in \mathrm{O}(p+1, q+1).$$

We note that Λ belongs to the identity-component of $\mathrm{O}(p+1, q+1)$, i.e. $\mathrm{SO}(p+1, q+1)$ if and only if Λ' is contained in the identity-component of $\mathrm{O}(p, q)$, i.e. $\mathrm{SO}(p, q)$. We define the conformal continuation associated to ϕ as

$$\begin{aligned} \hat{\phi} :& N^{p,q} \to N^{p,q} \\ \left(\xi^0 : \ldots : \xi^{n+1}\right) &\longmapsto \left(\xi^0 : \Lambda' \tilde{\xi} : \xi^{n+1}\right), \end{aligned}$$

where we denoted $\tilde{\xi} = \left(\xi^1 : \ldots : \xi^n\right)$. We verify that $\hat{\phi}$ satisfies eq. (1.9) by calculating

$$\begin{aligned} \hat{\phi}\left(\imath(x)\right) &= \left(\frac{1-x^2}{2} : \Lambda' x : \frac{1+x^2}{2}\right) \\ &= \left(\frac{1-(\Lambda' x)^2}{2} : \Lambda' x : \frac{1+(\Lambda' x)^2}{2}\right) \\ &= \imath\left(\phi(x)\right), \end{aligned}$$

where we have used the orthogonality of Λ'.

Translations. Let $\phi(x) = x + c$ with $c \in \mathbb{R}^{p,q}$. We define

$$\begin{aligned} \hat{\phi} :& N^{p,q} \to N^{p,q} \\ \left(\xi^0 : \ldots : \xi^{n+1}\right) &\longmapsto \left(\xi^0 - \tilde{\xi}\cdot c - \xi^+ c^2 : \tilde{\xi} + 2\xi^+ c : \xi^{n+1} + \tilde{\xi}\cdot + \xi^+ c^2\right) \end{aligned}$$

with $\xi^\pm := \frac{1}{2}\left(\xi^{n+1} \pm \xi^0\right)$ and calculate

$$\begin{aligned} \hat{\phi}\left(\imath(x)\right) &= \left(\frac{1-x^2}{2} - x\cdot c - \frac{c^2}{2} : x + c : \frac{1+x^2}{2} + x\cdot c + \frac{c^2}{2}\right) \\ &= \left(\frac{1-(x+c)^2}{2} : x + c : \frac{1+(x+c)^2}{2}\right) \\ &= \imath\left(\phi(x)\right). \end{aligned}$$

We can find an expression for the associated matrix $\Lambda \in \mathrm{O}(p+1, q+1)$ by considering the relation $\hat{\phi}(\xi) = \gamma\left(\Lambda\tilde{\xi}\right)$:

$$\Lambda_c = \begin{pmatrix} 1 - \frac{c^2}{2} & -c^T\eta & -\frac{c^2}{2} \\ c & \mathbb{1}_n & c \\ \frac{c^2}{2} & c^T\eta & 1 + \frac{c^2}{2} \end{pmatrix},$$

where η is the matrix representation of the metric $(\eta)_{pq} \equiv g_{p,q}$. A simple calculation show that $\Lambda_t \in \mathrm{O}(p+1, q+1)$ for all $t \in \mathbb{R}$. Furthermore, the map $\gamma : [0,1] \to \mathrm{O}(p+1, q+1)$ defined by $t \mapsto \Lambda_{tc}$ is a continuous path inside $\mathrm{O}(p+1, q+1)$ connecting Λ_c to the identity. Thus, Λ_c lies in the identity-component $\mathrm{SO}(p+1, q+1)$ of the orthogonal transformations.

Dilatations. Let $\phi(x) = \lambda x$ with $\lambda \in \mathbb{R}^+$. We define

$$\begin{aligned}
\hat{\phi} : N^{p,q} &\to N^{p,q} \\
\left(\xi^0 : \ldots : \xi^{n+1}\right) &\longmapsto \left(\frac{1+\lambda^2}{2\lambda}\xi^0 + \frac{1-\lambda^2}{2\lambda}\xi^{n+1} : \tilde{\xi} : \frac{1+\lambda^2}{2\lambda}\xi^{n+1} + \frac{1-\lambda^2}{2\lambda}\xi^0\right) \\
&= \left(\xi^+ + \frac{\lambda^2}{2}(\xi^0 - \xi^{n+1}) : \lambda\tilde{\xi} : \xi^+ - \frac{\lambda^2}{2}(\xi^0 - \xi^{n+1})\right).
\end{aligned}$$

With $\iota(x)^+ = \frac{1}{2}$ and $\iota(x)^- = \frac{x^2}{2}$ we calculate

$$\begin{aligned}
\hat{\phi}(\iota(x)) &= \left(\frac{1}{2} - \frac{\lambda^2}{2}x^2 : \lambda x : \frac{1}{2} + \frac{\lambda^2}{2}x^2\right) \\
&= \iota\left(\phi(x)\right).
\end{aligned}$$

Again, using the relation $\hat{\phi}(\xi) = \gamma(\Lambda_\lambda \xi)$ we receive an explicit form of the corresponding element in the orthogonal group

$$\Lambda_\lambda = \begin{pmatrix} \frac{1+\lambda^2}{2\lambda} & 0 & \frac{1-\lambda^2}{2\lambda} \\ 0 & \mathbb{1}_n & 0 \\ \frac{1-\lambda^2}{2\lambda} & 0 & \frac{1+\lambda^2}{2\lambda} \end{pmatrix}.$$

Analogously to the case of translations, we can verify that $\Lambda_\lambda \in \mathrm{SO}(p+1, q+1)$.

Special conformal transformations. Let $b \in \mathbb{R}^{p,q}$ and (cf.eq. (1.6))

$$\phi(x) = \frac{x - x^2 b}{1 - 2x\cdot b + x^2 b^2}, \qquad \text{for } x \in M_1 \subset \mathbb{R}^{p,q}. \tag{1.10}$$

We define

$$\begin{aligned}
\hat{\phi} : &N^{p,q} \to N^{p,q} \\
\left(\xi^0 : \ldots : \xi^{n+1}\right) &\mapsto \left(\xi^0 - \tilde{\xi}\cdot b + \xi^- b^2 : \tilde{\xi} - 2\xi^- b : \xi^{n+1} - \tilde{\xi}\cdot b + \xi^- b^2\right).
\end{aligned}$$

To simplify the calculation we denote the denominator in eq. (1.10) as $N := 1 - 2x\cdot b + x^2 b^2$. We find that

$$\left(\phi(x)\right)^2 = \frac{1}{N^2}\left(x^2 + (x^2)^2 b^2 - 2x^2(x\cdot b)\right) = \frac{x^2}{N}.$$

As before, we prove that $\hat{\phi}$ is indeed the conformal continuation of ϕ by calculating

$$\begin{aligned}
\hat{\phi}(\iota(x)) &= \left(\frac{1-x^2}{2} - x\cdot b + \frac{x^2}{2}b^2 : x - x^2 b : \frac{1+x^2}{2} - x\cdot b + \frac{x^2}{2}b^2\right) \\
&= \left(\frac{N - x^2}{2} : x - x^2 b : \frac{N + x^2}{2}\right) \\
&= \left(\frac{1 - \frac{x^2}{N}}{2} : \frac{x - x^2 b}{N} : \frac{1 - \frac{x^2}{N}}{2}\right) = \iota\left(\phi(x)\right).
\end{aligned}$$

The matrix generating this transformation is given by

$$\Lambda_b = \begin{pmatrix} 1 - \frac{b^2}{2} & -b^T\eta & \frac{b^2}{2} \\ b & \mathbb{1}_n & -b \\ -\frac{b^2}{2} & -b^T\eta & 1 + \frac{b^2}{2} \end{pmatrix} \qquad \in \mathrm{SO}(p+1, q+1).$$

We have proven that every conformal transformation ϕ has a conformal continuation $\hat{\phi} : \xi \mapsto \gamma(\Lambda\xi)$, with $\Lambda \in \mathrm{SO}(p+1, q+1)$. It can be shown that the map $\phi \mapsto \hat{\phi}$ is injective. Conversely, for every conformal map $\Psi : N^{p,q} \to N^{p,q}$ there exists a nonempty open subset $U \subset \imath(\mathbb{R}^{p,q})$ such that $V := \Psi(U)$ is again open and nonempty. Then, the map

$$\phi := \imath^{-1} \circ \Psi \circ \imath : \; \imath^{-1}(U) \to \imath^{-1}(V)$$

is conformal, since is has a conformal continuation $\hat{\phi}$, which must be equal to Ψ. Together with theorem 1.4, we conclude that the group of conformal continuations is isomorphic to $\mathrm{O}(p+1, q+1)\backslash\{\pm 1\}$ and thus

$$\mathrm{Conf}(\mathbb{R}^{p,q}) \cong \mathrm{SO}(p+1, q+1).$$

□

1.3 The Conformal Group of $\mathbb{S}^1 \times \mathbb{R}$

In string theory, the manifold under consideration is a cylinder embedded in a D-dimensional MINKOWSKI space. More precisely, this manifold is $\mathbb{S}^1 \times \mathbb{R}$ equipped with the MINKOWSKI metric. In contrast to the conformal compactification of $\mathbb{R}^{p,q}$ in the previous section, the cylinder $\mathbb{S}^1 \times \mathbb{R}$ acts as a conformal compactification of itself as we will show in the following.

First, we introduce light-cone coordinates via the mapping

$$\begin{aligned} \imath : \mathbb{S}^1 \times \mathbb{R} &\longrightarrow \mathbb{R}^2/2\pi\mathbb{Z}, \\ \begin{pmatrix} x \\ t \end{pmatrix} &\longmapsto \begin{pmatrix} x_+ \\ x_- \end{pmatrix} := \begin{pmatrix} x+t \\ x-t \end{pmatrix}. \end{aligned}$$

It is invertible, since for $(x_+, x_-) := \imath(x, t)$

$$\imath^{-1}\begin{pmatrix} x_+ \\ x_- \end{pmatrix} := \frac{1}{2}\begin{pmatrix} x_+ + x_- \\ x_+ - x_- \end{pmatrix} \overset{!}{=} \begin{pmatrix} x \\ t \end{pmatrix}$$

does not depend on the representative of the equivalence class where (x_+, x_-) is in. This is due to the invariance of t under simultaneous translations of x_+ and x_- and the fact that $x \in \mathbb{S}^1$.

Lemma 1.1. *The map* $\imath$ *is an isometry – and thus conformal with conformal factor* $\Omega = 1$ *– when we equip the space* $\mathbb{R}^2/2\pi\mathbb{Z}$ *with the metric*

$$\eta = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

Proof. We verify that $\imath$ satisfies the definition of a conformal transformation (cf. eq. (1.1)) with $\Omega = 1$:

$$\begin{aligned} \eta_{ij}\partial_\mu \imath^i \partial_\nu \imath^j &= \frac{1}{2}\partial_\mu \imath^+ \partial_\nu \imath^- + \frac{1}{2}\partial_\mu \imath^- \partial_\nu \imath^+ \\ &= \frac{1}{2}(-1)^{\nu+1} + \frac{1}{2}(-1)^{\mu+1} = g_{\mu\nu}. \end{aligned}$$

□

Our goal is to show that the conformal transformations we found in theorem 1.3 are exactly the global conformal transformations in $\mathbb{R}^2/2\pi\mathbb{Z}$ with the metric η. We first need to adapt said theorem to the light-cone coordinates. Let us begin with the last two conditions:

$$\begin{aligned} (\partial_+ + \partial_-)\frac{1}{2}(\Phi_+ + \Phi_-) &= \partial_0 u = \pm\partial_1 v = \pm(\partial_+ - \partial_-)\frac{1}{2}(\Phi_+ - \Phi_-), \\ (\partial_+ - \partial_-)\frac{1}{2}(\Phi_+ + \Phi_-) &= \partial_1 u = \pm\partial_0 v = \pm(\partial_+ + \partial_-)\frac{1}{2}(\Phi_+ - \Phi_-). \end{aligned}$$

Adding up the equations resp. subtracting them from each other, yields

$$\partial_+ (\Phi_+ + \Phi_-) = \pm \partial_+ (\Phi_+ - \Phi_-)\,,$$
$$\partial_- (\Phi_+ + \Phi_-) = \mp \partial_+ (\Phi_+ - \Phi_-)\,.$$

The case of the upper (lower) sign implies that $\Phi_\pm$ has no dependence on $x_\mp$ ($x_\pm$). Then, the first condition becomes

$$\begin{aligned} & u_0^2 > v_0^2 \\ \Longleftrightarrow\ & ((\partial_+\partial_-)(\Phi_+ + \Phi_-))^2 > (\pm 1)^2 ((\partial_+\partial_-)(\Phi_+ - \Phi_-))^2 \\ \Longleftrightarrow\ & 2(\partial_+ + \partial_-)\Phi_+ (\partial_+ + \partial_-)\Phi_- > -2(\partial_+ + \partial_-)\Phi_+ (\partial_+ + \partial_-)\Phi_- \\ \Longleftrightarrow\ & \Phi'_+\Phi'_- > 0. \end{aligned}$$

Thus, we either have $f', g' > 0$ or $f', g' < 0$. As for the transformations of the Minkowski plane, we again have four connected components of diffeomorphisms on $\mathbb{R} \times \mathbb{S}^1$. These are given by the identity component (1), the time inversions (T), the parity transformations (P), and inversions of both time and space (TP). Transformations with a positive Jacobian determinant – transformations contained in 1 and TP – are orientation-preserving, whereas those with negative Jacobian determinant – those inside S and T – are orientation-inverting. We conclude that all conformal orientation-preserving transformations of $\mathbb{R} \times \mathbb{S}^1$ are of the form[1]

$$\Phi : \mathbb{R}^2/2\pi\mathbb{Z} \longrightarrow \mathbb{R}^2/2\pi\mathbb{Z}$$
$$\begin{pmatrix} x_+ \\ x_- \end{pmatrix} \longmapsto \begin{pmatrix} f(x_+) \\ g(x_-) \end{pmatrix}$$

for some functions $f, g \in (\mathrm{Diff}_+(\mathbb{R}) \times \mathrm{Diff}_+(\mathbb{R})) \cup (\mathrm{Diff}_-(\mathbb{R}) \times \mathrm{Diff}_-(\mathbb{R}))$. Here and in the following, $\mathrm{Diff}_\pm(M)$ denotes the group of orientation-preserving resp. orientation-inverting diffeomorphisms on the manifold M. In general, such a function is not well-defined without f and g satisfying

$$\begin{pmatrix} f(x_+ + 2\pi) \\ g(x_- + 2\pi) \end{pmatrix} = \begin{pmatrix} f(x_+) + 2\pi\, n_\Phi \\ g(x_-) + 2\pi\, n_\Phi \end{pmatrix}.$$

with $n_\Phi \in \mathbb{Z}$.

Theorem 1.6. *The conformal orientation-preserving diffeomorphisms on* $\mathbb{R} \times \mathbb{S}^1$ *are given by*

$$\Phi : \mathbb{R}^2/2\pi\mathbb{Z} \longrightarrow \mathbb{R}^2/2\pi\mathbb{Z}, \quad \begin{pmatrix} x_+ \\ x_- \end{pmatrix} \longmapsto \begin{pmatrix} f(x_+) \\ g(x_-) \end{pmatrix}$$

where $f, g \in \left(\mathrm{Diff}_+(\mathbb{S}^1) \times \mathrm{Diff}_+(\mathbb{S}^1)\right) \cup \left(\mathrm{Diff}_-(\mathbb{S}^1) \times \mathrm{Diff}_-(\mathbb{S}^1)\right)$.

Proof. We readily check that Φ suffices the definition of a conformal transformation (cf. eq. (1.1)):

$$\begin{aligned} \eta_{ij}\partial_\mu\Phi^i\partial_\nu\Phi^j &= \frac{1}{2}\partial_\mu\Phi^+\partial_\nu\Phi^- + \frac{1}{2}\partial_\mu\Phi^-\partial_\nu\Phi^+ \\ &= \frac{1}{2}\delta^+_\mu\delta^-_\nu f'\cdot g' + \frac{1}{2}\delta^-_\mu\delta^+_\nu f'\cdot g' \\ &= \eta_{\mu\nu}\underbrace{f'\cdot g'}_{>0}. \end{aligned}$$

[1]The conformal orientation-inverting transformations can be obtained by inverting the dependencies of f and g on x_+ and x_-. They are not inside the connected component containing the identity and thus of no interest to the study of the conformal group.

For Φ to be a diffeomorphism, we have to ensure that its inverse is well-defined. By definition, the function f is injective. We want Φ to be injective as well, i.e. that $\Phi(x) = \Phi(y)$ implies $x = y + 2\pi m$ for some $m \in \mathbb{Z}$. Since f satisfies $f(x_+ + 2\pi) = f(x_+) + 2\pi n$ for some $n \in \mathbb{Z}$, we have

$$\begin{aligned} \Phi(x_+, x_-) &= \Phi(y_+, y_-) \\ \Rightarrow \exists j \in \mathbb{Z}: \quad f(x_+) &= f(y_+) + 2\pi j \\ \overset{!}{\Longleftrightarrow} f(y_+) + 2\pi m n &= f(y_+) + 2\pi j. \end{aligned}$$

Thus, if we would like to be able to find such an $m \in \mathbb{Z}$ for any $j \in \mathbb{Z}$, i.e. that Φ is injective, n must be 1. We conclude, f and g are 2π-periodic if and only if Φ is a conformal diffeomorphism. □

We derived that the conformal group – the identity component of the group of all orientation-preserving conformal transformations – of the cylinder $\mathbb{R} \times \mathbb{S}^1$ is isomorphic to

$$\mathrm{Conf}(\mathbb{R} \times \mathbb{S}^1) \cong \mathrm{Diff}_+(\mathbb{S}^1) \times \mathrm{Diff}_+(\mathbb{S}^1).$$

Chapter 2

Central Extensions and Quantisation of Symmetries

This section yields the mathematical prerequisites for the extension of the Witt algebra in chapter 3 and for the quantisation of symmetries in general. We will start off by introducing the concept of a central extension by applying it to groups. Before adapting the formalism to Lie algebras, we explain the need of a central extension when quantising a system that is invariant under a certain symmetry group.

2.1 Central Extensions of Groups

In the following, the number 1 represents the trivial group, i.e. the group containing only the identity.

Definition 2.1. An *extension* of a group G by the group A is a short exact sequence, i.e. a sequence of group homomorphisms

$$1 \longrightarrow A \xrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow 1,$$

such that the kernel of each group homomorphism is equal to the image of the previous homomorphism. This is equivalent to requiring ι to be injective, π to be surjective, and $\mathrm{Im}(\iota) = \ker(\pi)$. An extension is called *central*, if the group A is abelian.

The simplest example is probably the trivial extension, given by

$$1 \longrightarrow A \xrightarrow{\iota} A \times G \xrightarrow{\pi_2} G \longrightarrow 1,$$

where $\iota(a) = (a, 1_G)$ and π_i denotes the projection onto the i-th coordinate-axis. This describes an extension since ι is injective, π is surjective, and

$$\mathrm{Im}(\iota) = \left\{ (a, \mathbb{1}_G) \mid a \in A \right\} = \ker(\pi).$$

The most important example for us will be the extension of the group of unitary operators $\mathrm{U}(\mathbb{H})$ acting on a Hilbert space $\mathbb{H}$. In the following we denote the inner product and the induced norm on $\mathbb{H}$ as $\langle \cdot | \cdot \rangle$ resp. $\| \cdot \|$. We define the corresponding projective space $\mathbb{P} = \mathbb{P}(\mathbb{H})$ as the vector space of all linear subspaces of $\mathbb{H}$, namely

$$\begin{aligned} \mathbb{P}(\mathbb{H}) &:= \left(H \setminus \{0\} \right) / \sim \\ f \sim g &\Longleftrightarrow \exists \lambda \in \mathbb{C} \setminus \{0\} : \quad f = \lambda g. \end{aligned}$$

The projective space corresponds to the physical space of states: two wavefunctions differing by a complex factor represent the same classical or physical state. As in section 1.2, we denote the quotient map by $\gamma : \mathbb{H}\backslash\{0\} \to \mathbb{P}$. The *transition probability* between two physical states $\phi = \gamma(f)$ and $\psi = \gamma(g)$ is given by

$$\delta(\phi, \psi) := \frac{|\langle f|g\rangle|^2}{\|f\|^2 \|g\|^2}$$

Definition 2.2. A *projective transformation* is a bijection $T : \mathbb{P} \to \mathbb{P}$, such that

$$\delta(T\phi, T\psi) = \delta(\phi, \psi), \quad \forall \phi, \psi \in \mathbb{P}.$$

The group of all these projective transformations is denoted by $\mathrm{Aut}(\mathbb{P})$, where the structure of the group is given by composition. A transformation of a system is a symmetry if and only if it leaves the transition probabilities invariant. By definition, $\mathrm{Aut}(\mathbb{P})$ is the complete symmetry group of the system.

We define a map $\hat{\gamma} : \mathrm{U}(\mathbb{H}) \to \mathrm{Aut}(\mathbb{P})$ via

$$\hat{\gamma}(U)(\phi) := \gamma(Uf), \quad \text{for } \phi = \gamma(f).$$

This mapping is well-defined, since we have for $\phi = \gamma(f)$ and $\psi = \gamma(g)$

$$\begin{aligned}
\delta\left(\hat{\gamma}(U)(\phi), \hat{\gamma}(U)(\psi)\right) &= \delta\left(\gamma(Uf), \gamma(Ug)\right) \\
&= \frac{|\langle Uf|Ug\rangle|^2}{\|Uf\|^2 \|Ug\|^2} \\
&= \frac{|\langle f|g\rangle|^2}{\|f\|^2 \|g\|^2} \\
&= \delta(\phi, \psi),
\end{aligned}$$

implying that $\hat{\gamma}(U) \in \mathrm{Aut}(\mathbb{P})$ for all $U \in \mathrm{U}(\mathbb{H})$.

Definition 2.3. We define the group of *unitary projective transformations* as the subgroup

$$\mathrm{U}(\mathbb{P}) := \hat{\gamma}(\mathrm{U}(\mathbb{H})) \subset \mathrm{Aut}(\mathbb{P}).$$

We find that the group of unitary projective transformations is a central extension of the group of unitary transformations of the Hilbert space $\mathbb{H}$.

Lemma 2.1. *The group* $\mathrm{U}(\mathbb{H})$ *is the central extension of* $\mathrm{U}(\mathbb{P})$ *by* $\mathrm{U}(1)$*, or equivalently:*

$$1 \longrightarrow \mathrm{U}(1) \xrightarrow{\iota} \mathrm{U}(\mathbb{H}) \xrightarrow{\hat{\gamma}} \mathrm{U}(\mathbb{P}) \longrightarrow 1$$

with $\iota(\lambda) = \lambda \mathbb{1}_{\mathbb{H}}$ *and* $\hat{\gamma}$ *as above, defines an exact sequence.*

Proof. First, we note that if this sequence is exact, this extension is central, since the group $\mathrm{U}(1)$ is abelian. Obviously, the map ι is injective and by definition 2.3 $\hat{\gamma}$ is surjective. It remains to show that $\ker \hat{\gamma} = \mathrm{U}(1)\mathbb{1}_{\mathbb{H}} = \mathrm{Im}(\iota)$, where the second equality is trivial.

"$\subset$" Let $U \in \ker\hat{\gamma}$. This means that U satisfies for all $\phi = \gamma(f) \in \mathbb{P}$

$$\hat{\gamma}(U)(\phi) = \phi = \gamma(f).$$

By definition of $\hat{\gamma}$, we have $\hat{\gamma}(u)(\phi) = \gamma(Uf)$ and thus $\gamma(f) = \gamma(Uf)$. Conclusively, there exists a factor $\lambda_f \in \mathbb{C}\backslash\{0\}$ with $\lambda_f f = Uf$. In fact, if such a factor exists, it has to be in U(1) for U to be unitary. We need to show that this factor λ_f is independent of the element f, namely $\lambda_f \equiv \lambda$, such that $U = \lambda \mathbb{1}_{\mathbb{H}}$. Linearity of U yields the relation

$$\lambda_f f + \lambda_g g = Uf + Ug = U(f+g) = \lambda_{f+g} f + \lambda_{f+g}.$$

We now take the inner product on both sides with f. If g lies in the complement of $\mathbb{C}f$, i.e. $\langle g|f\rangle = 0$, then

$$\lambda_f \langle f|f\rangle = \lambda_{f+g} \langle f|f\rangle \quad \Rightarrow \quad \lambda_{f+g} = \lambda_f.$$

On the other hand, if $g = \mu f$, then

$$\lambda_{\mu f} \mu f = U(\mu f) = \mu U f = \mu \lambda_f f \quad \Rightarrow \quad \lambda_{\mu f} = \lambda_f$$

and we conclude that $U = \lambda \mathbb{1}_{\mathbb{H}} \in \mathrm{U}(1)\mathbb{1}_{\mathbb{H}}$.

"$\supset$" Let $U = \lambda \mathbb{1}_{\mathbb{H}}$ with $\lambda \in \mathrm{U}(1)$ and $\phi = \gamma(f) \in \mathbb{P}$. Then

$$\hat{\gamma}(U)(\phi) \equiv \gamma(Uf) = \gamma(\lambda f) = \gamma(f) = \phi \quad \Rightarrow \hat{\gamma}(U) = \mathbb{1}_P.$$

Thus, $\mathrm{U}(1)\mathbb{1}_{\mathbb{H}} \subset \ker(\hat{\gamma})$.

□

2.2 Quantisation of Symmetries

Suppose we have a system Y with a symmetry group G. What we actually mean by that, is that there exists a group homomorphism $\tau : G \to \mathrm{Aut}(Y)$ mapping each element of the symmetry group to an automorphism that leaves the physical laws of the system invariant. After the quantisation, one assumes that the map τ induces a new group homomorphism $T : G \to \mathrm{U}(\mathbb{P})$. However, since we are mostly interested in the HILBERT space $\mathbb{H}$, we need to develop a consistent method of adapting the symmetry group G to $\mathbb{H}$ resp. $\mathrm{U}(\mathbb{H})$.

Definition 2.4. The set containing

$$W_f(U_0, r) := \{U \in \mathrm{U}(\mathbb{H}) \mid \|U_0(f) - U(f)\| < r\}$$

for all $f \in \mathbb{H}$, $U_0 \in \mathrm{U}(\mathbb{H})$ and $r \in \mathbb{R}^+$ is a subbasis of a topology on $\mathrm{U}(\mathbb{H})$, which we call the *strong (operator) topology*.

This means that – by definition – a subset $W \subset U(\mathbb{H})$ is open if and only if for every element $U_0 \in W$ there exist finitely many $W_{f_j}(U_0, r_j)$ such that

$$U_0 \in \bigcap_{j=1}^{N} W_{f_j}(U_0, r_j) \subset W.$$

Definition 2.5. A group G equipped with a topology is called a *topological group* if the group operation and the inversion are continuous according to that topology.

Definition 2.6. A *unitary* representation of a topologial group G is a continuous homomorphism

$$\rho : G \longrightarrow \mathrm{U}(\mathbb{H})$$

with respect to the strong operator topology.

Definition 2.7. A *projective* representation of a topologial group G is a continuous homomorphism

$$\rho : G \longrightarrow \mathrm{U}(\mathbb{P})$$

with respect to the strong operator topology.

As we already mentioned in the beginning of this section, we want to raise the question what happens with a symmetry of a system $T : G \longrightarrow \mathrm{U}(\mathbb{P})$ after quantisation. In other words, can we find a unitary transformation $S : G \longrightarrow \mathrm{U}(\mathbb{H})$, such that the following diagram is commutative?

$$\begin{array}{ccccccccc} & & & & & & G & & \\ & & & & & \overset{S}{\swarrow} & \downarrow T & & \\ 1 & \longrightarrow & \mathrm{U}(1) & \xrightarrow[\iota]{} & \mathrm{U}(\mathbb{H}) & \xrightarrow[\hat{\gamma}]{} & \mathrm{U}(\mathbb{P}) & \longrightarrow & 1 \end{array}$$

No, such a *lift* of a symmetry group G does not exist in general. But, as we will discuss in the following theorem, a similar relation can be achieved with the help of a central extension of G.

Theorem 2.1. *Let G be a symmetry group endowed with a homomorphism $T : G \to \mathrm{U}(\mathbb{P})$. Then, we can lift this symmetry via a central extension E of G by $\mathrm{U}(1)$, meaning that there exists a homomorphism $S : E \to \mathrm{U}(\mathbb{H})$, such that*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{U}(1) & \xrightarrow{\iota_1} & E & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ & & \mathbb{1} \downarrow & & S \downarrow & & \downarrow T & & \\ 1 & \longrightarrow & \mathrm{U}(1) & \xrightarrow[\iota_2]{} & \mathrm{U}(\mathbb{H}) & \xrightarrow[\hat{\gamma}]{} & \mathrm{U}(\mathbb{P}) & \longrightarrow & 1 \end{array}$$

commutes.

Proof. Let

$$E := \left\{ (U, g) \in \mathrm{U}(\mathbb{H}) \times G \,|\, \hat{\gamma}(U) = T(g) \right\}.$$

Since $\hat{\gamma}$ and T are homomorphisms, it is easy to verify that E is a subgroup of $\mathrm{U}(H) \times G$. We define

$$\begin{aligned} \iota &: \lambda \longmapsto (\lambda \mathbb{1}_{\mathbb{H}}, \mathbb{1}_G), \\ \pi &: = \pi_2 \end{aligned}$$

and find that this defines a central extension of G. Finally, we set

$$S := \pi_1$$

and deduce that S satisfies for all elements $(U, g) \in E$

$$\begin{aligned} (T \circ \pi)(U, g) &= T(g) \\ &= \hat{\gamma}(U) \\ &= (\hat{\gamma} \circ S)(U, g). \end{aligned}$$

This proves that the diagram is indeed commutative. □

Unfortunately, theorem 2.1 does not tell us whether the homomorphism S is continuous – and thus a unitary representation of G – or not. To make such a statement, we first need the following lemma.

Lemma 2.2. *The group* $\mathrm{U}(\mathbb{H})$ *is a topological group with respect to the strong operator topology.*

Proof. We need to verify that both the group operation and the inversion are continuous with respect to the strong topology. We start with the inversion.

Let $U \in \mathrm{U}(\mathbb{H})$ and $V \in W_g(U,r)$ with $g \in \mathbb{H}$ and $r \in R^+$. Then with $f := U(g)$ we have

$$\begin{aligned} \left\| U^{-1}(f) - V^{-1}(f) \right\| &= \left\| g - V^{-1}(U(g)) \right\| \\ &= \left\| V(g) - U(g) \right\| \\ &< r. \end{aligned}$$

Thus $V^{-1} \in W_{U^{-1}g}(U^{-1}, r)$ and the inversion is continuous with respect to the strong topology.

Now let $U, U' \in \mathrm{U}(\mathbb{H})$, $VV' \in W_g(UU', r)$ with $g \in \mathbb{H}$ and $r \in \mathbb{R}^+$. With $f := U'(g)$ it follows that

$$\begin{aligned} \left\| UU'(g) - VV'(g) \right\| &= \left\| UU'(g) - VV'(g) + VU'(g) - VU'(g) \right\| \\ &\leq \left\| UU'(g) - VU'(g) \right\| + \left\| VU'(g) - VV'(g) \right\| \\ &= \left\| U(f) - V(f) \right\| + \left\| U'(g) - V'(g) \right\| \end{aligned}$$

Requiring that $V \in W_{U'^{-1}f}(U, \frac{r}{2})$ and $V' \in W_g(U', \frac{r}{2})$ guarantees that $VV' \in W_g(UU', r)$ and finally yields that the group composition is continuous with respect to the strong topology. ☐

On the group of unitary projective transformations $\mathrm{U}(\mathbb{P}) = \hat{\gamma}(\mathrm{U}(\mathbb{H}))$, we define a topology via

$$V \subset \mathrm{U}(\mathbb{P}) \text{ open} \iff \hat{\gamma}^{-1}(V) \subset \mathrm{U}(\mathbb{H}) \text{ open}.$$

With this so-called quotient topology, $\mathrm{U}(\mathbb{P})$ is also a topological group.

Corollary 2.1. *Let G, T, E and S be as in theorem 2.1. If G is a topological group, and T is a projective representation, then the group homomorphism S is a unitary representation in* $\mathrm{U}(\mathbb{H})$.

Proof. As in the proof of theorem 2.1, we choose $E = \left\{(U,g) \in \mathrm{U}(\mathbb{H}) \times G \,|\, \hat{\gamma}(U) = Tg\right\}$. Since G is a topological group, we can equip E with the product topology of $\mathrm{U}(\mathbb{H}) \times G$. Thus, E is a topological group and $S := \pi_1$ is trivially continuous. ☐

We want to end the discussion of extensions of groups by defining the equivalence of central extensions. This will come in handy in section 3.1 where we will develop *the* central extension of the Witt algebra over $\mathbb{C}$.

Definition 2.8. We call two extensions E and E' of G over A *equivalent*, if they are isomorph via ψ such that the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{U}(1) & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \Big\downarrow \mathbb{1} & & \Big\downarrow \psi & & \Big\downarrow \mathbb{1} & & \\ 1 & \longrightarrow & \mathrm{U}(1) & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

commutes.

Definition 2.9. We say that an extension

$$1 \longrightarrow A \xrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow 1$$

splits, if there exists a homomorphism $\sigma : G \to E$ with $\pi \circ \sigma = \mathbb{1}_G$.

Lemma 2.3. *A central extension is equivalent to the trivial central extension if and only if it splits.*

Proof. Suppose we have a central extension

$$1 \longrightarrow A \xrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow 1$$

that is equivalent to the central extension. By definition, there exists an isomorphism $\psi : A \times G \to E$ such that $\pi \circ \psi = \pi_2 \mathbb{1}_G = \pi_2$. Defining

$$\begin{aligned} \sigma : G &\longrightarrow E, \\ g &\longmapsto \psi(\mathbb{1}_A, g) \end{aligned}$$

yields for all $g \in G$

$$\begin{aligned} (\pi \circ \sigma)(g) &= (\pi \circ \psi)(\mathbb{1}_A, g) \\ &= \pi_2(\mathbb{1}_A, g) \\ &= g. \end{aligned}$$

Thus, the central extension splits.

On the other hand, if the extension splits, there exists a homomorphism $\sigma : G \to E$ with $\pi \circ \sigma = \mathbb{1}_G$. We define the isomorphism

$$\begin{aligned} \psi : A \times G &\longrightarrow E, \\ (a, g) &\longmapsto \iota(a)\sigma(g). \end{aligned}$$

Then, for all $(a, g) \in A \times G$ and $\alpha \in A$ we have

$$\begin{aligned} (\pi \circ \psi)(a, g) &= \pi(\iota(a)\sigma(g)) \\ &= \pi(\iota(a))\pi(\sigma(g)) \\ &= 1_G g \\ &= (\mathbb{1}_G \circ \pi_2)(a, g), \end{aligned}$$

$$\begin{aligned} (\psi \circ \iota_{\text{triv}})(\alpha) &= \psi(\alpha, \mathbb{1}_G) \\ &= \iota(\alpha) \\ &= (\iota \circ \mathbb{1}_A)(\alpha). \end{aligned}$$

We have shown that the corresponding diagram commutes. Therefore, this extension is equivalent to the trivial extension. □

2.3 Central Extensions of LIE Algebras

Similarly to the case of groups, we can define extensions for LIE algebras in the following way. We will denote the LIE algebra with only one element by 0.

Definition 2.10. Let $\mathfrak{a}$, $\mathfrak{g}$ and $\mathfrak{e}$ be LIE algebras over a field $\mathbb{K}$. Then $\mathfrak{e}$ is an *extension* of $\mathfrak{g}$ by $\mathfrak{a}$, if there is an exact sequence (cf. definition 2.1) of LIE algebra homomorphisms

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} g \longrightarrow 0.$$

The extension is called *central* if $\mathfrak{a}$ is abelian.

We want to present two examples that will be of high importance for our later discussion. The first one is the central extension of the HEISENBERG algebra, given by

$$\mathrm{H} := \mathbb{C}\left[T, T^{-1}\right] \oplus \mathbb{C}Z,$$

the direct sum of the LAURENT polynomials and the central element Z. The LIE -bracket is defined as

$$[f \oplus \lambda Z, g \oplus \mu Z] := \sum_{k \in \mathbb{Z}} k f_k g_{-k} Z,$$

with $f = \sum f_n T^n$ and $g = \sum g_n T^n$. To build the bridge to our previous discussion, we note that this extension comes from the exact sequence

$$0 \longrightarrow \mathbb{C} \xrightarrow{\iota} \mathrm{H} \xrightarrow{\pi} \mathbb{C}\left[T, T^{-1}\right] \longrightarrow 0,$$

where we defined the two maps

$$\begin{aligned} \iota : \mathbb{C} &\longrightarrow \mathrm{H} & \pi : \mathrm{H} &\longrightarrow \mathbb{C}\left[T, T^{-1}\right] \\ \lambda &\longmapsto \lambda Z, & f \otimes \lambda Z &\longmapsto f. \end{aligned}$$

A convenient basis for H is

$$\left\{a_n := T^n \mid n \in \mathbb{Z}\right\} \cup Z,$$

satisfying the commutation relations of the quantum harmonic oscillator

$$[a_m, a_n] = m\delta_{m+n} Z \qquad [Z, a_m] = 0,$$

for all $m, n \in \mathbb{Z}$.

The second example we would like to mention, is the *affinisation* of a LIE algebra $\mathfrak{g}$, defined by

$$\hat{\mathfrak{g}} := \mathfrak{g}\left[T, T^{-1}\right] \oplus \mathbb{C}Z,$$

where $\mathfrak{g}\left[T, T^{-1}\right] = \mathbb{C}\left[T, T^{-1}\right] \otimes \mathfrak{g}$. The KILLING form on $\mathfrak{g}$ is given by

$$\begin{aligned} \kappa : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathbb{C} \\ (a, b) &\longmapsto \operatorname{tr}\left(\operatorname{ad}_a \circ \operatorname{ad}_b\right) \end{aligned}$$

and satisfies

$$\begin{aligned}\kappa\big([a,b],c\big) &= \mathrm{tr}\big(\mathrm{ad}_{[a,b]}\circ\mathrm{ad}_c\big)\\ &= \mathrm{tr}\big([\mathrm{ad}_a,\mathrm{ad}_b]\circ\mathrm{ad}_c\big)\\ &= \mathrm{tr}\big(\mathrm{ad}_a\circ\mathrm{ad}_b\circ\mathrm{ad}_c-\mathrm{ad}_b\circ\mathrm{ad}_a\circ\mathrm{ad}_c\big)\\ &= \mathrm{tr}\big(\mathrm{ad}_a\circ\mathrm{ad}_b\circ\mathrm{ad}_c-\mathrm{ad}_a\circ\mathrm{ad}_c\circ\mathrm{ad}_b\big)\\ &= \kappa\big(a,[b,c]\big),\end{aligned}$$

where we have used the cyclic property of the trace and that the LIE bracket on End($\mathfrak{g}$) is given by $[f,g] = f\circ g - g\circ f$. The affinisation of $\mathfrak{g}$ is equipped with the LIE bracket

$$\begin{aligned}\left[T^m\otimes a,\, T^n\otimes b\right] &:= T^{m+n}\otimes[a,b]+m\kappa(a,b)\delta_{m+n}Z\\ \left[T^m\otimes a,\, Z\right] &:= 0.\end{aligned}$$

Redefining the elements of this vector space as $f_m := T^m\otimes f$ yields

$$[a_m, b_n] = [a,b]_{m+n}+m\kappa(a,b)\delta_{m+n}Z.$$

To verify that this is indeed a central extension of $\mathfrak{g}\left[T,T^{-1}\right]$ we have to find LIE algebra homomorphisms ι and π, such that

$$0\longrightarrow\mathbb{C}\stackrel{\iota}{\longrightarrow}\hat{\mathfrak{g}}\stackrel{\pi}{\longrightarrow}\mathfrak{g}\left[T,T^{-1}\right]\longrightarrow 0$$

satisfies definition 2.10. For example

$$\begin{aligned}\iota:\mathbb{C}&\longrightarrow\hat{\mathfrak{g}} &\qquad \pi:\hat{\mathfrak{g}}&\longrightarrow\mathfrak{g}\left[T,T^{-1}\right],\\ \lambda&\longmapsto\lambda Z, & f\otimes a+\lambda Z&\longmapsto f\otimes a.\end{aligned}$$

Note that if $\mathfrak{g}=\mathbb{C}$, then we recover the previous example of the central extension of theHEISENBERG algebra.

Definition 2.11. We say that an extension

$$0\longrightarrow\mathfrak{a}\stackrel{\iota}{\longrightarrow}\mathfrak{e}\stackrel{\pi}{\longrightarrow}\mathfrak{g}\longrightarrow 0$$

splits, if there exists a LIE algebra homomorphism $\sigma:\mathfrak{g}\to\mathfrak{e}$ with $\pi\circ\sigma=\mathbb{1}_{\mathfrak{g}}$.

Similar to the case of group extensions, we can define *equivalence* and *triviality* of LIE algebra extensions. Furhermore, lemma 2.3 is valid in exactly the same way.

Lemma 2.4. *A LIE algebra extension splits if and only if it is trivial.*

Definition 2.12. A *2-cocycle* on a LIE algebra $\mathfrak{g}$ with values in $\mathfrak{a}$ is a map $\theta:\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{a}$ that satisfies:

i) It is bilinear and alternating.

ii) $\theta\big(X,[Y,Z]\big)+\theta\big(Y,[Z,X]\big)+\theta\big(Z,[X,Y]\big)=0.$

With this definition, we can define two subspaces of the space $\mathrm{Alt}^2(\mathfrak{g},\mathfrak{a})$ of alternating bilinear maps from $\mathfrak{g}\times\mathfrak{g}$ to $\mathfrak{a}$ as

$$\begin{aligned}Z^2(\mathfrak{g},\mathfrak{a}) &:= \left\{\theta\in\mathrm{Alt}^2(\mathfrak{g},\mathfrak{a})\,|\,\theta \text{ satisfies condition ii)}\right\}\\ B^2(\mathfrak{g},\mathfrak{a}) &:= \left\{\theta\in Z^2(\mathfrak{g},\mathfrak{a})\,|\,\exists\mu\in\mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{a}):\theta(X,Y)=\mu([X,Y])\right\}.\end{aligned}$$

Definition 2.13. The *second cohomology group* of $\mathfrak{g}$ with values in $\mathfrak{a}$ is given by

$$H^2(\mathfrak{g},\mathfrak{a}) := Z^2(\mathfrak{g},\mathfrak{a}) \big/ B^2(\mathfrak{g},\mathfrak{a}) .$$

We want to show that $H^2(\mathfrak{g},\mathfrak{a})$ is isomorph to the set of equivalence classes of central extensions of $\mathfrak{g}$ by $\mathfrak{a}$. Let us suppose that there is a central extension

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0.$$

We note that for every Lie algebra extension of this form, there is a map $\beta : \mathfrak{g} \to \mathfrak{e}$ with $\pi \circ \beta = \mathbb{1}_{\mathfrak{g}}$. Nevertheless, this function does not need to be a homomorphism. A measure of "how much it deviates" from being a homomorphism can be defined as

$$\begin{aligned} \theta : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathfrak{a}, \\ (X, Y) &\longmapsto \left[\beta(X), \beta(Y)\right] - \beta([X, Y]) . \end{aligned} \tag{2.1}$$

Obviously, this map satisfies condition i) – bilinearity and alternativity – of definition 2.12. Additionally, a tedious but elementary calculation shows that θ also fulfils condition ii).

Lemma 2.5. *With θ as in eq.* (2.1) *we find that*

i) Every central extension $\mathfrak{e}$ originates from such a θ.

ii) Every θ induces a central extension $\mathfrak{e}$.

Proof.

i) We have $\mathfrak{e} \cong \mathfrak{g} \oplus \mathfrak{a}$ as vector spaces, since the linear map

$$\psi : \mathfrak{g} \times \mathfrak{a} \to \mathfrak{e}, \quad X \oplus Y \longmapsto \beta(X) + Y$$

is invertible with $\pi \circ \beta = \mathbb{1}_{\mathfrak{g}}$. For a general Lie bracket on $\mathfrak{e}$, we can define a function θ implicitly via

$$\left[X \oplus Z, Y \oplus Z'\right]_{\mathfrak{e}} =: [X, Y]_{\mathfrak{g}} + \theta(X, Y), \qquad \text{for } X, Y \in \mathfrak{g},\ Z, Z' \in \mathfrak{a}.$$

This is well-defined, since $\mathfrak{a}$ – as a subalgebra of $\mathfrak{e}$ – is in the center of $\mathfrak{e}$. From the definition of θ is follows that it satisfies both conditions in definition 2.12. We can rewrite the last equation as

$$\left[\beta(X), \beta(Y)\right] = \left[\beta(X) + Z, \beta(Y) + Z'\right] = \beta([X, Y]) + \theta(X, Y)$$

to get the explicit form of θ that agrees with eq. (2.1).

ii) On the other hand, if we were given such a function θ, we can just define a Lie bracket on $\mathfrak{e} \cong \mathfrak{g} \oplus \mathfrak{a}$ as

$$\left[X \oplus Z, Y \oplus Z'\right]_{\mathfrak{e}} := [X, Y]_{\mathfrak{g}} + \theta(X, Y), \qquad \text{for } X, Y \in \mathfrak{g},\ Z, Z' \in \mathfrak{a}.$$

With θ satisfying the two conditions of definition 2.12, it is clear that this defines a central extension $\mathfrak{e}$ of $\mathfrak{g}$ by $\mathfrak{a}$.

□

Now that we examined the relation between central extensions and 2-cocycles, we can prove the following corollary.

Corollary 2.2. *A central extension splits – and thus is trivial – if and only if its corresponding 2-cocycle θ is in $B^2(\mathfrak{g},\mathfrak{a})$, i.e. if there exists a homomorphism $\mu \in \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{a})$ such that $\theta(X,Y) = \mu([X,Y])$.*

Proof.

"⇒" Suppose the central extension splits, i.e. there is a Lie algebra homomorphism $\sigma : \mathfrak{g} \to \mathfrak{e}$ with $\pi \circ \sigma = \mathbb{1}_{\mathfrak{g}}$. This implies that there exists a map $\mu \in \mathrm{Hom}(\mathfrak{g},\mathfrak{a})$ with $\sigma(X) = X + \mu(X)$ for all $X \in \mathfrak{g}$. With the Lie bracket on $\mathfrak{e} \cong \mathfrak{g} \oplus \mathfrak{a}$ defined as in section 2.3, we have

$$[\sigma(X),\sigma(Y)]_{\mathfrak{e}} = [X,Y]_{\mathfrak{g}} + \theta(X,Y).$$

On the other hand, since we assumed that σ is a Lie algebra homomorphism, is satisfies

$$[\sigma(X),\sigma(Y)]_{\mathfrak{e}} = \sigma\left([X,Y]_{\mathfrak{g}}\right) = [X,Y]_{\mathfrak{g}} + \mu\left([X,Y]_{\mathfrak{g}}\right).$$

We conclude that $\theta(X,Y) = \mu([X,Y])$ and thus $\theta \in B^2(\mathfrak{g},\mathfrak{a})$.

"⇐" If there is a map $\mu \in \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{a})$ with $\theta(X,Y) = \mu([X,Y])$, then θ automatically suffices both conditions in definition 2.12. To show that there exists a splitting map, we define $\sigma : \mathfrak{g} \to \mathfrak{a}$ as $\sigma(X) := X + \mu(X)$ and calculate

$$\begin{aligned}\sigma\left([X,Y]_{\mathfrak{g}}\right) &= [X,Y]_{\mathfrak{g}} + \mu([X,Y]) \\ &= [X,Y]_{\mathfrak{g}} + \theta(X,Y) \\ &= [X,Y]_{\mathfrak{e}} \\ &= \left[X + \mu(X), Y + \mu(Y)\right]_{\mathfrak{e}} \\ &= [\sigma(X),\sigma(Y)]_{\mathfrak{e}},\end{aligned}$$

which proves that $\sigma : \mathfrak{g} \to \mathfrak{e}$ is Lie algebra homomorphism and since $\pi \circ \sigma = \mathbb{1}_{\mathfrak{g}}$ it is a splitting map.

□

We summarise our findings in the following theorem.

Theorem 2.2. *The equivalence classes of central extensions of $\mathfrak{g}$ by $\mathfrak{a}$ are in one-to-one correspondence to the second cohomology group of $\mathfrak{g}$ with values in $\mathfrak{a}$*

$$H^2(\mathfrak{g},\mathfrak{a}) = Z^2(\mathfrak{g},\mathfrak{a}) \big/ B^2(\mathfrak{g},\mathfrak{a}).$$

Chapter 3

The Virasoro Algebra

We found in section 1.3 that the conformal group of the cylinder $\mathbb{R} \times \mathbb{S}^1$ is isomorphic to $\mathrm{Diff}_+(\mathbb{S}) \times \mathrm{Diff}_+(\mathbb{S})$. In this section we will examine a subalgebra of the complexification of the Lie algebra of $\mathrm{Diff}_+(\mathbb{S}^1)$ – called the Witt algebra –, develop its central extension to the Virasoro algebra, and study the representations of the Virasoro algebra.

Let $f_t : \mathbb{S}^1 \to \mathbb{S}^1$ be a family of orientation-preserving diffeomorphisms on the circle, with $f_0 = \mathbb{1}_{\mathbb{S}^1}$. The Lie algebra of the Lie group $\mathrm{Diff}_+(\mathbb{S}^1)$ is given by the elements of the form

$$a(x) := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f_t(x) \qquad \in \mathrm{T}_{\mathbb{1}}\mathrm{Diff}_+(\mathbb{S}^1).$$

Conversely, we can expand the family f_t for small values of t around the identity and find

$$f_t(x) = x + a(x)\,t + \mathcal{O}(t^2), \tag{3.1}$$

The differential of the conjugation in $\mathrm{Diff}_+(\mathbb{S}^1)$ at the identity, i.e. of $\Psi_f : \mathrm{Diff}_+(\mathbb{S}^1) \to \mathrm{Diff}_+(\mathbb{S}^1)$ with $\Psi_f(g) := f \circ g \circ f^{-1}$, yields a diffeomorphism $\mathrm{Ad}(f) := \left(\mathrm{d}\Psi_f\right)_e : \mathrm{T}_{\mathbb{1}}\mathrm{Diff}_+(\mathbb{S}^1) \to \mathrm{T}_{\mathbb{1}}\mathrm{Diff}_+(\mathbb{S}^1)$ on the tangent space of the identity, where

$$\begin{aligned} \mathrm{Ad} : \mathrm{Diff}_+(\mathbb{S}^1) &\longrightarrow \mathrm{Diff}\big(\mathrm{T}_{\mathbb{1}}\mathrm{Diff}_+(\mathbb{S}^1)\big), \\ f_t &\longmapsto \left(\frac{\mathrm{d}g_s}{\mathrm{d}t}\bigg|_{s=0} \mapsto f_t \circ \frac{\mathrm{d}g_s}{\mathrm{d}s}\bigg|_{s=0} \circ f_t^{-1} \right) \end{aligned}$$

is the adjoint representation of the group. Taking the differential of the adjoint representation, yields a map

$$\begin{aligned} \mathrm{ad} : \mathrm{T}_{\mathbb{1}}\mathrm{Diff}_+(\mathbb{S}^1) &\longrightarrow \mathrm{End}\big(\mathrm{T}_{\mathbb{1}}\mathrm{Diff}_+(\mathbb{S}^1)\big), \\ \frac{\mathrm{d}f_t}{\mathrm{d}t}\bigg|_{t=0} &\longmapsto \left(\frac{\mathrm{d}g_s}{\mathrm{d}t}\bigg|_{s=0} \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \left(f_t \circ g_s \circ f_t^{-1}\right) \right). \end{aligned}$$

This is well-defined, since, in first order in t, we have for $f_t(x) = x + v(x)t$ and $g_s(x) = x + w(x)s$ [1]

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \left(f_t \circ g_s \circ f_t^{-1}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \Big[x - v(x)t + w\big(x - v(x)t\big)s + v\Big(x - v(x)t + w\big(x - v(x)t\big)s\Big)t\Big] \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Big[w\big(x - v(x)t\big) + v\Big(w\big(x + v(x)t\big)t\Big)\Big] \\ &= v(w(x)) - w(v(x)) \\ &= \frac{\mathrm{d}f_t}{\mathrm{d}t}\bigg|_{t=0} \circ \frac{\mathrm{d}g_s}{\mathrm{d}s}\bigg|_{s=0} - \frac{\mathrm{d}g_s}{\mathrm{d}s}\bigg|_{s=0} \circ \frac{\mathrm{d}f_t}{\mathrm{d}t}\bigg|_{t=0}. \end{aligned}$$

[1] We assume that for small t and s we can write $v(x + yt) = v(x) + v(y)t + \mathcal{O}(t^2)$ and analogous for w. This is motivated by the interpretation of v and w as vector fields, acting linearly on the elements of the tangent space.

We define the LIE bracket on $\mathrm{Diff}_+(\mathbb{S}^1)$ as

$$[X, Y] := (\mathrm{ad}(X))\,(Y).$$

We already claim that we should identify the LIE algebra of $\mathrm{Diff}_+(\mathbb{S}^1)$ with the vector fields on the circle $\mathrm{Vect}(\mathbb{S}^1)$. The vector fields are defined via

$$v(x) := -\frac{\partial f_t(x)}{\partial t}\Big|_{t=0}\frac{\mathrm{d}}{\mathrm{d}x},$$

for some familiy $f_t : \mathbb{S}^1 \to \mathbb{S}^1$ of orientation-preserving diffeomorphisms with $f_0 = \mathbb{1}_{\mathbb{S}^1}$. They act *linearly* on the space of diffeomorphisms on the circle via

$$\left(v\phi\right)(x) := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\left(\phi\left(f_t^{-1}\right)\right)(x), \quad \text{for} \quad \phi \in \mathrm{Diff}_+(\mathbb{S}^1). \tag{3.2}$$

Using the chain rule, we find (in accordance with the definition of v)

$$\begin{aligned}\left(v\phi\right)(x) &= \phi'\left(f_0^{-1}(x)\right)\frac{\partial f_t^{-1}(x)}{\partial t}\Big|_{t=0}\\ &= -\phi'(x)\frac{\partial f_t(x)}{\partial t}\Big|_{t=0}.\end{aligned}$$

Taking $\phi = \mathbb{1}_{\mathbb{S}^1}$, we are led to expand the map f_t in powers of t via

$$f_t(x) = \mathbb{1} - v(x)\,t + \mathcal{O}(t^2). \tag{3.3}$$

Comparing both expansions of f_t in terms of the LIE algebra (cf. eq. (3.1)) resp. of the vector fields (cf. eq. (3.3)) allows us to set

$$\mathrm{LIE}\left(\mathrm{Diff}_+(\mathbb{S}^1)\right) \equiv -\mathrm{Vect}(\mathbb{S}^1),$$

where the LIE bracket on the vector fields is simply given by $[X, Y](z) = X(Y(z) - Y(X(z))$ for $z \in \mathbb{S}^1$.

3.1 WITT algebra and its Extension to the VIRASORO algebra

As we mentioned before, we would like to study the WITT algebra

$$\mathrm{W} := \mathrm{span}_{\mathbb{C}}\left\{L_n | n \in \mathbb{Z}\right\} \subset \mathrm{Vect}^{\mathbb{C}}(\mathbb{S}^1) := \mathrm{Vect}(\mathbb{S}^1) \otimes \mathbb{C},$$

where the L_ns are defined for $z = \exp(\imath\theta)$, $\theta \in \mathbb{R}$

$$L_n := z^{1-n}\frac{\mathrm{d}}{\mathrm{d}z} = z^{1-n}\underbrace{\frac{\mathrm{d}\theta}{\mathrm{d}z}}_{=-\imath z^{-1}}\frac{\mathrm{d}}{\mathrm{d}\theta} = -\imath\exp(-\imath\theta n)\frac{\mathrm{d}}{\mathrm{d}\theta} \in \mathrm{Vect}^{\mathbb{C}}(\mathbb{S}^1). \tag{3.4}$$

Indeed, we readily verify that this defines a LIE algebra. For $f \in C^\infty(\mathbb{S}^1, \mathbb{C})$ we have

$$\begin{aligned}[L_m, L_n]\,f &= z^{1-m}\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{1-n}\frac{\mathrm{d}}{\mathrm{d}z}f\right) - z^{1-n}\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{1-m}\frac{\mathrm{d}}{\mathrm{d}z}f\right)\\ &= z^{1-m-n}\left((1-n)-(1-m)\right)\frac{\mathrm{d}}{\mathrm{d}z}f\\ &= (m-n)L_{m+n}f.\end{aligned}$$

Theorem 3.1. *Every non-trivial central extension of* W *by* $\mathbb{C}$ *– up to equivalence (cf. definition 2.8) – is given by a complex scalar multiple of the 2-cocycle*

$$\omega(L_m, L_n) = \delta_{n+m}\frac{m}{12}(m^2-1).$$

This theorem implies for the second cohomology class of W with values in $\mathbb{C}$ – which we introduced in definition 2.13 – that

$$H^2(\mathrm{W},\mathbb{C}) \cong \mathbb{C}.$$

Proof. We need to show that $\omega \in Z^2(\mathrm{W},\mathbb{C})\backslash B^2(\mathrm{W},\mathbb{C})$ and that every $\theta \in Z^2(\mathrm{W},\mathbb{C})$ is equivalent to a scalar multiple of ω.

$\underline{\omega \in Z^2(\mathrm{W},\mathbb{C})}$: The first condition in definition 2.12 – bilinearity and alternativity – is by definition satisfied. For the second requirement we calculate

$$\begin{aligned}
&\omega\big(L_i,[L_j,L_k]\big)+\omega\big(L_j,[L_k,L_i]\big)+\omega\big(L_k,[L_i,L_j]\big)\\
&\qquad= (j-k)\,\omega(L_i,L_{j+k})+(k-i)\,\omega(L_j,L_{k+i})+(i-j)\,\omega(L_k,L_{i+j})\\
&\qquad\propto \delta_{i+j+k}\big((j-k)\,i(i^2-1)+(k-i)\,j(j^2-1)+(i-j)\,k(k^2-1)\big)\\
&\qquad= (2j+i)\,i(i^2-1)-(2i+j)\,j(j^2-1)-(i-j)(i+j)\big((i+j)^2-1\big)\\
&\qquad= 2ji^3-2ji+i^4-i^2-2ij^3+2ij-j^4+j^2-i^4-2i^3j-i^2j^2\\
&\qquad\qquad +i^2+i^2j^2+2ij^3+j^4-j^2\\
&\qquad= 0.
\end{aligned}$$

$\underline{\omega \notin B^2(\mathrm{W},\mathbb{C})}$: Let ω be contained in $B^2(\mathrm{W},\mathbb{C})$. By definition, there exists a $\mu \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{W},\mathbb{C})$ such that ω can be written as $\omega(X,Y)=\mu([X,Y])$. Let $n \in \mathbb{N}$, then

$$\frac{m}{12}(m^2-1)=\omega(L_m,L_n)=\mu([L_m,L_n])=2m\,\mu(L_0).$$

Thus μ cannot be well defined.

<u>**Uniqueness:**</u> It remains to show that every element $\theta \in Z^2(\mathrm{W},\mathbb{C})$ lies in the same equivalence class of $H^2(\mathrm{W},\mathbb{C})$ as ω. This means that for every $\theta \in Z^2(\mathrm{W},\mathbb{C})$ we need to find $\lambda \in \mathbb{C}, \mu \in B^2(\mathrm{W},\mathbb{C})$ such that

$$\theta(X,Y)=\lambda\omega(X,Y)-\mu([X,Y]).$$

We start by using the second condition of a 2-cocycle (cf. definition 2.12) on θ and find with $k=0$

$$\begin{aligned}
0 &= \theta\big(L_i,[L_j,L_k]\big)+\theta\big(L_j,[L_k,L_i]\big)+\theta\big(L_k,[L_i,L_j]\big)\\
&= j\,\theta(L_i,L_j)-i\,\theta(L_j,L_i)+(i-j)\,\theta(L_0,L_{i+j})\\
&= -(i+j)\,\theta(L_i,L_j)+(i-j)\,\theta(L_0,L_{i+j}),
\end{aligned}$$

$$\Longrightarrow \quad \theta(L_m,L_n)=\frac{m-n}{m+n}\theta(L_0,L_{m+n}) \quad \text{for} \quad m+n\neq 0. \tag{3.5}$$

This allows us to define the homomorphism $\mu \in \mathrm{Hom}(\mathrm{W},\mathbb{C})$

$$\begin{aligned}
\mu(L_m) &:= \frac{1}{m}\theta(L_0,L_m) \quad \text{for} \quad m\neq 0,\\
\mu(L_0) &:= -\frac{1}{2}\theta(L_1,L_{-1}).
\end{aligned}$$

We claim that the map

$$\begin{aligned}
\Theta: \mathrm{W}\times\mathrm{W} &\longrightarrow \mathbb{C},\\
(X,Y) &\longmapsto \theta(X,Y)+\mu([X,Y])
\end{aligned}$$

is a scalar multiple of the map ω from the theorem. As we mentioned in the beginning, this would prove the theorem.

First, we should ensure that Θ is a 2-cocycle as defined in definition 2.12. Obviously, it is bilinear and alternating due to the fact that θ is a 2-cocycle, and that μ is linear in the LIE bracket, which is bilinear and alternating by definition. Since the LIE bracket satisfies the JACOBI identity, Θ furthermore satisfies the second condition and thus is indeed a 2-cocycle.

Secondly, we show that $\Theta(L_m, L_n)$ is proportional to δ_{m+n}. Let $m \neq -n$

$$\begin{aligned}\Theta(L_m, L_n) &= \theta(L_m, L_n) + \mu([L_m, L_n]) \\ &= \frac{m-n}{m+n}\theta(L_0, L_{m+n}) - (m-n)\mu(L_{m+n}) \\ &= 0,\end{aligned}$$

where we used eq. (3.5) and the definition of μ. This justifies the Ansatz

$$\Theta(L_m, L_n) = h(n)\,\delta_{m+n},$$

where h needs to be odd, i.e. $h(-n) = -h(n)$. We use the second cocycle condition with $i + j + k = 0$ and $j = 1$ to derive the recurrence relation

$$\begin{aligned}0 &= \Theta\big(L_i, [L_j, L_k]\big) + \Theta\big(L_j, [L_k, L_i]\big) + \Theta\big(L_k, [L_i, L_j]\big) \\ &= (j-k)\,\Theta(L_i, L_{j+k}) + (k-i)\,\Theta(L_j, L_{k+i}) + (i-j)\,\Theta(L_k, L_{i+j}) \\ &= (2j-i)\,h(i) - (2i-j)\,h(j) - (i-j)\,h(i+j) \\ &= (2-i)\,h(i) - (2i-1)h(1) - (i-1)\,h(i+1).\end{aligned}$$

This expression simplifies even more due to the vanishing of $h(1)$:

$$\begin{aligned}h(1) &= \Theta(L_1, L_{-1}) \\ &= \theta(L_1, L_{-1}) + \mu([L_1, L_{-1}] \\ &= \theta(L_1, L_{-1}) + 2\mu(L_0) \\ &= \theta(L_1, L_{-1}) + 2\left(-\frac{1}{2}\theta(L_1, L_{-1})\right) \\ &= 0.\end{aligned}$$

Thus, the proportionality factor h of Θ has to satisfy for all $i \neq 1$

$$h(i+1) = \frac{i+2}{i-1}h(i).$$

The solution to this recurrence relation is given by

$$h(i) = \frac{h(2)}{6}\,i(i^2 - i) \quad \text{for} \quad i \in \mathbb{N}, \tag{3.6}$$

which we prove by induction: For the case $i = 2$ we find

$$\begin{aligned}h(2+1) &\equiv \frac{2+2}{2-1}h(2) = 4\,h(2) \\ h(3) &\equiv \frac{h(2)}{6}\,3(3^2 - 1) = 4\,h(2).\end{aligned}$$

For the induction step, we set $i+1>2$ and assume that $h(i)$ is given by eq. (3.6). It follows that

$$\begin{aligned} h(i+1) &= \frac{i+2}{i-1} h(i) \\ &= \frac{i+2}{i-1} \frac{h(2)}{6} i(i^2-1) \\ &= \frac{h(2)}{6} (i+1) i (i+2) \\ &= \frac{h(2)}{6} (i+1) \big((i+1)^2 - 1\big). \end{aligned}$$

We conclude that eq. (3.6) is valid for $i \in \mathbb{N}$ and that any $\theta \in Z^2(\mathrm{W}, \mathbb{C})$ can be written as

$$\theta(X,Y) = \Theta(X,Y) - \mu([X,Y]) = \lambda\omega(X,Y) - \mu([X,Y]),$$

with $\lambda = 2h(2)$. This proves the theorem. □

Definition 3.1. The central extension generated by the 2-cocycle ω (cf. lemma 2.5) is called the *VIRASORO algebra*.

In the context of vector spaces, we thus have

$$\mathrm{Vir} = \mathrm{W} \oplus \mathbb{C}Z,$$

where the LIE bracket is given by

$$[L_m, L_n]_{\mathrm{Vir}} = [L_m, L_n]_{\mathrm{W}} + \omega(L_m, L_n) Z = (m-n) L_{m+n} + \frac{m}{12}(m^2-1)\delta_{m+n} Z.$$

Together with the previous section, where we elaborated the importance of central extensions for the process of quantisation, we are led to investigate the VIRASORO algebra and its representations.

3.2 Representation Theory of the VIRASORO Algebra

Now that we have developed a central extension of the WITT algebra, we are interested in how it should act on a complex vector space.

Definition 3.2. A *cyclic vector* of a representation $\rho : \mathrm{Vir} \to \mathrm{End}_{\mathbb{C}}(V)$ is an element $v_0 \in V$, such that

$$V = \mathrm{span}\left\{\rho(X_1)\ldots\rho(X_k)v \,|\, X_i \in \mathrm{Vir},\ 0 \le i \le k,\ k \in \mathbb{N}\right\}.$$

Definition 3.3. A *highest-weight representation* is a representation $\rho : \mathrm{Vir} \to \mathrm{End}_{\mathbb{C}}(V)$ with a cyclic vector v_0, such that there exist $c, h \in \mathbb{C}$ with

$$\begin{aligned} \rho(Z) v_0 &= c v_0, \\ \rho(L_0) v_0 &= h v_0, \\ \rho(L_n) v_0 &= 0, \quad \text{for } n \in \mathbb{N}. \end{aligned}$$

The cyclic vector v_0 of a highest-weight representation is sometimes called *highest-weight vector* or *vacuum vector*.

Definition 3.4. A *VERMA module* for $c, h \in \mathbb{C}$ is a vector space $M(c,h)$ with basis

$$\mathscr{B} := \{v_0\} \cup \{\rho(L_{-n_1}) \dots \rho(L_{-n_k}) v \,|\, 0 < n_k \leq \dots \leq n_1,\ k \in \mathbb{N}\},$$

where v_0 is a highest-weight vector of a highest-weight representation $\rho : \mathrm{Vir} \to \mathrm{End}_C(M(c,h))$.

Lemma 3.1. *There exists a VERMA module for any values of $c, h \in \mathbb{C}$.*

Proof. To show this statement, we will give an explicit description of this module. First, we define the vector space spanned by the basis

$$\{v_0\} \cup \{v_{n_1 \dots n_k} \,|\, 0 < n_k \leq \dots \leq n_1,\ k \in \mathbb{N}\}.$$

On this vector space we can define the representation $\rho : \mathrm{Vir} \to \mathrm{End}_{\mathbb{C}}(M(c,h))$ by

$$\begin{aligned}
\rho(Z) &:= c \mathbb{1}_{M(c,h)}, \\
\rho(L_0) v_0 &:= h v_0, \\
\rho(L_0) v_{n_1 \dots n_k} &:= \left(\sum_{i=1}^{k} n_i + h \right) v_{n_1 \dots n_k}, \\
\rho(L_n) v_0 &:= \begin{cases} 0 & \text{for } n \geq 1 \\ v_{-n} & \text{for } n \leq -1 \end{cases}, \\
\rho(L_{-n}) v_{n_1 \dots n_k} &:= v_{n n_1 \dots n_k} \quad \text{for} \quad n \geq n_1.
\end{aligned}$$

The remaining relations are defined via the LIE bracket on the VIRASORO algebra. One can show that this definition yields a representation by verifying that we have for each basis element w

$$\left[\rho(L_m), \rho(L_n)\right] w = \rho\left([L_m, L_n]\right) w.$$

□

Let us suppose we have a VIRASORO module V for $c, h \in \mathbb{C}$. We can define subspaces of V by $V_0 := \mathbb{C} v_0$ and

$$V_N := \mathrm{span}\{\rho(L_{-n_1}) \dots \rho(L_{-n_k}) v_0 \mid 0 < n_k \leq \dots \leq n_1,\ \sum_j n_j = N,\ k \in \mathbb{N}\}.$$

We want to show that V_N is an eigenspace of $\rho(L_n)$ with the eigenvalue $N + h$. Thus, let us assume that $X = \rho(L_{-n_1}) \dots \rho(L_{-n_k}) v_0 \in V_N$. With $\sum_j n_j = N$ we have

$$\begin{aligned}
\rho(L_0) X &= \rho(L_0) \rho(L_{-n_1}) \dots \rho(L_{-n_k}) v_0 \\
&= \underbrace{\left[\rho(L_0), \rho(L_{-n_1})\right]}_{=\rho([L_0, L_{-n_1}])} \rho(L_{-n_2}) \dots \rho(L_{-n_k}) v_0 + \rho(L_{-n_1}) \rho(L_0) \rho(L_{-n_2}) \dots \rho(L_{-n_k}) v_0 \\
&= n_1 \rho(L_{-n_1}) \dots \rho(L_{-n_k}) v_0 + \rho(L_{-n_1}) \rho(L_0) \rho(L_{-n_2}) \dots \rho(L_{-n_k}) v_0 \\
&\;\;\vdots \\
&= \left(\sum_{j=1}^{k} n_j \right) \rho(L_{-n_1}) \dots \rho(L_{-n_k}) v_0 + \rho(L_{-n_1}) \dots \rho(L_{-n_k}) \rho(L_0) v_0 \\
&= (N + h) X.
\end{aligned}$$

Therefore, we can decompose the vector space V into the direct sum

$$V = \bigoplus_{j=0}^{\infty} V_N.$$

Lemma 3.2. *For any submodule*[2] *U of a Virasoro module V we have*

$$U = \bigoplus_{j=0}^{\infty} (V_N \cap U).$$

Proof. As we mentioned before, the VIRASORO module can be decomposed into a direct sum of eigenspaces V_N with respect to $\rho(L_0)$. Therefore, for every element $u = u_0 \oplus \ldots \oplus u_m \in U \subset V$ with $u_j \in V_j$ we receive by repeatedly applying $\rho(L_0)$ to u

$$\begin{aligned} u &= u_0 \oplus \ldots \oplus u_m \\ \rho(L_0)u &= hu_0 \oplus \ldots \oplus (m+h)u_m \\ &\vdots \\ \rho(L_0)^{m-1}u &= h^{m-1}u_0 \oplus \ldots \oplus (m+h)^{m-1}u_m. \end{aligned}$$

This is a system of m linear independent equations, yielding for every u_j a linear combination of $u,\ \rho(L_0)u,\ \ldots,\ \rho(L_0)^{m-1}u$ that is equal to u_j. Thus $u_j \in U$ and especially $u_j \in V_j \cap U$. □

Suppose that V is a complex vector space with a positive semi-definite hermitian form $H : V \times V \to \mathbb{C}$. The real infinitesimal generators of unitary transformations must be skew-symmetric operators. In the case of a representation $\rho : \mathrm{Vir} \to \mathrm{End}_{\mathbb{C}}(V)$ of the VIRASORA algebra this would mean that the elements[3] (cf. eq. (3.4))

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta} &= \imath L_0, \\ \cos(n\theta)\frac{\mathrm{d}}{\mathrm{d}\theta} &= -\frac{\imath}{2}(L_n + L_{-n}), \\ \sin(n\theta)\frac{\mathrm{d}}{\mathrm{d}\theta} &= -\frac{1}{2}(L_n - L_{-n}) \end{aligned}$$

have to satisfy $H\big(\rho(D)v, w\big) = -H\big(v, \rho(D)w\big)$ for all $v, w \in V$ and $D \in \big\{\mathrm{d}/\mathrm{d}\theta,\ \cos(n\theta)\mathrm{d}/\mathrm{d}\theta,\ \sin(n\theta)\mathrm{d}/\mathrm{d}\theta\big\}$. We can implement this condition using the following definition.

Definition 3.5. We call a representation $\rho : \mathrm{Vir} \to \mathrm{End}_{\mathbb{C}}(V)$ *unitary*, if

$$\begin{aligned} H\big(\rho(L_n)v, w\big) &= H\big(v, \rho(L_{-n})w\big), \\ H\big(\rho(Z)v, w\big) &= H\big(v, \rho(Z)w\big) \end{aligned}$$

for all $v, w \in V$ and $n \in \mathbb{Z}$.

First, let us try to introduce a hermitian form on a VERMA module $M(c,h)$ for $c, h \in \mathbb{R}$. We know that every vector $u \in M(c,h)$ has a unique representation $u = \bigoplus_{j=0}^{\infty} u_j$ with $u_j \in V_j$. Since the subspace V_0 is one dimensional

[2] This means that U is a subspace of V with $\rho(X)U \subset U$ for all $X \in \mathrm{Vir}$.

[3] Every element of $\mathrm{Vect}(\mathbb{S}^1)$ is of the form $F = f\frac{\mathrm{d}}{\mathrm{d}\theta}$ where f must be contained in $C^{\infty}(\mathbb{R})$ and 2π-periodic. Thus we can write f as a FOURIER series with generators $\cos(n\theta)$ and $\sin(n\theta)$.

with the basis $\{v_0\}$, there exists a $\lambda \in \mathbb{C}$ such that $u_0 = \lambda v_0$. We define the *expectation value* of this vector as exactly this proportionality factor λ

$$\langle u \rangle := \lambda.$$

We are now ready to construct a hermitian form on the basis elements of $M(c,h)$. We define the map $H : \mathscr{B} \times \mathscr{B} \to \mathbb{C}$ via[4]

$$H\left(v_{n_1 \dots n_k}, v_{m_1 \dots m_l}\right) := \left\langle L_{n_k} \dots L_{n_1} v_{m_1 \dots m_l} \right\rangle = \left\langle L_{n_k} \dots L_{n_1} L_{-m_1} \dots L_{-m_l} v_0 \right\rangle.$$

This definition automatically yields $H(v_0, v_0) = 1$ and $H(v_0, v_{n_1 \dots n_k}) = H(v_{n_1 \dots n_k}, v_0) = 0$. We note that this map is real and symmetric. The former is an immediate implication of $c, h \in \mathbb{R}$, whereas the latter requires applying repeatedly the commutation relations of L_n (or more precisely of $\rho(L_n)$). We can continue this symmetric map to a hermitian form on $M(c,h)$ by defining for $u = \sum_{i \in \mathbb{N}_0} \lambda_i u_i$ and $u' = \sum_{i \in \mathbb{N}_0} \mu_i u'_i$ with $u_i, u'_i \in \mathscr{B}$ for all $i \in N_0$

$$H(w, w') := \sum_{i,j \in \mathbb{N}_0} \overline{\lambda_i} \mu_j H(u_i, u'_j).$$

Corollary 3.1. *The hermitian form H we defined above, is the unique hermitian form on $M(c,h)$ with $c, h \in \mathbb{R}$ that satisfies for all $u, u' \in M(c,h), n \in \mathbb{Z}$*

$$H(L_n v, w) = H(v, L_{-n} w) \quad H(Z v, w) = H(v, Z w) \quad H(v_0, v_0) = 1.$$

Proof. Obviously, the hermitian form H from above fulfils these conditions. For a general hermitian form $\tilde{H}$ satisfying the theorem we have that $\tilde{H}(v_0, v_{n_1 \dots n_k}) = \tilde{H}(L_{n_1} v_0, v_{n_2 \dots n_k}) = 0$. Therefore, the expression

$$\tilde{H}(v_{n_1 \dots n_k}, v_{m_1 \dots m_l}) = \tilde{H}(v_0, L_{n_k} \dots L_{n_1} v_{m_1 \dots m_l})$$

can only depend on c, h and $\tilde{H}(v_0, v_0)$. □

Corollary 3.2. *The eigenspaces of L_0 are pairwise orthogonal with respect to the hermitian form H.*

Proof. Let us assume that $u \in V_n$ and $w \in V_m$ with $n > m$. Then u is given by a linear combination of basis elements $v_{n_1 \dots n_k}$ with $\sum_j n_j = n$ and w by $v_{m_1 \dots m_l}$ satisfying $\sum_j m_j = m$. We have

$$H(u, w) = \left\langle L_{n_k} \dots L_{n_1} L_{-m_1} \dots L_{-m_l} v_0 \right\rangle.$$

The commutation relations of the VIRASORO operators and $n > m$ allow us to bring the operator acting on v_0 into the form PL_s with $s > 0$ and some operator P. The statement follows with $L_n v_0 = 0$ for $n > 0$. □

Corollary 3.3. *A positive semi-definite hermitian form H implies $c, h \geq 0$.*

Proof. Let $n \in \mathbb{N}$. Then

$$\begin{aligned} 0 &\leq H(v_n, v_n) \\ &= H(v_0, L_n L_{-n} v_0) \\ &= 2nh + \frac{n}{12}(n^2 - 1)c \end{aligned}$$

implies for $n = 1$ that $h \geq 0$. If $c < 0$, there would exist an $n \in \mathbb{N}$ such that $H(v_n, v_n) < 0$. □

We merely want to mention that one can show unitarity of a VERMA module $M(c,h)$ – i.e. positive definiteness of the corresponding hermitian form H – for $c > 1$ and $h > 0$. The proof of this statement includes the derivation of the KAC determinant and would exceed the scope of this work.

[4] From now on, we will refer to the representation implicitly, i.e. L_n acting on a vector v in the representation space means $\rho(L_n)v$.

Last but not least, we would like to investigate reducibility of the representation $M(c,h)$. As a reminder, a representation is *decomposable*, if there exist invariant proper subspaces $V,W \subset M(c,h)$ with $V \oplus W = M(c,h)$. A weaker statement is *reducibility*, implying that there simply *is* an invariant proper subspace. We finish this section with the following corollary.

Corollary 3.4. *Let $c,h \in \mathbb{C}$. The VERMA module $M = M(c,h)$ satisfies:*

i) M is indecomposable.

ii) If M is a unitary representation, then it is also irreducible.

Proof.

i) Let us suppose that there are invariant proper subspaces $V,W \subset M$ with $V \oplus W = M$. With lemma 3.2 we have
$$V = \bigoplus_{j=0}^{\infty} \left(M_j \cap V\right) \quad \text{and} \quad W = \bigoplus_{j=0}^{\infty} \left(M_j \cap W\right).$$
Since $\dim(M_0) = 1$ we either have $M_0 \cap V = \{0\}$ or $M_0 \cap V = M_0$ with the analogous result for W. Since V and W are supposed to be orthogonal and sum up the vector space M, the subspace M_0 is contained in either V or W. Without loss of generality, we assume $M_0 \cap V = M_0$. Due to the invariance under the action of L_n, V makes up the whole vector space M and neither V nor W are proper subspaces.

ii) Let V be an invariant proper subspace. We define
$$V^{\perp} := \left\{ w \in M \,|\, \forall v \in V : \; H(w,v) = 0 \right\}.$$
This subspace of M is invariant, due to the positive definiteness of H and
$$H(u, L_n v) = H(L_{-n} u, v) = 0$$
if $v \in V^{\perp}$ and $u \in V$. Positive definiteness of H implies that V and $V^{\perp}$ are orthogonal and furthermore $V \oplus V^{\perp} = M$. The statement follows with i).

□

Chapter 4

Vertex Algebras

4.1 Formal Distributions

Definition 4.1. A *formal distribution* on a vector space V over the field $\mathbb{C}$ is a formal series in the indeterminates $Z = (z_1, \dots z_n)$

$$A(z_1, \dots, z_n) = \sum_{j \in \mathbb{Z}^n} A_j z^j \equiv \sum_{j \in \mathbb{Z}^n} A_{j_1 \dots j_n} z_1^{j_1} \dots z_n^{j_n}, \quad A_j \in V.$$

We will denote the vector space of formal distributions on V in Z by $V\left[\left[Z^{\pm}\right]\right] \equiv V\left[\left[z_1^{\pm}, \dots, z_n^{\pm}\right]\right]$.

For the case with only one indeterminate, we call $\mathrm{Res}_z A(z) := A_{-1}$ the *residue* of $A(z)$. Naturally, we define the *formal derivative* as the map $\partial_z : V\left[\left[z^{\pm}\right]\right] \to V\left[\left[z^{\pm}\right]\right]$ and

$$\partial_z \left(\sum_{j \in \mathbb{Z}} A_j z^j \right) := \sum_{j \in \mathbb{Z}} (j+1) A_{j+1} z^j.$$

Lemma 4.1. *Let $A \in V\left[\left[z^{\pm}\right]\right]$ be a formal distribution in one indeterminate. The map from the* Laurent *polynomials $\mathbb{C}\left[z^{\pm}\right]$ to the vector field V given by*

$$\begin{aligned} \Phi_A : \mathbb{C}\left[z^{\pm}\right] &\longrightarrow V, \\ f(z) &\longmapsto \mathrm{Res}_z A(z) f(z), \end{aligned}$$

is linear. Furthermore, it yields an isomorphism between $V\left[\left[z^{\pm}\right]\right]$ and $\mathrm{Hom}\left(\mathbb{C}\left[z^{\pm}\right], V\right)$.

In retrospect, this property of A – i.e. assigning an element in the original vector space to a function – motivates the name formal *distribution*.

Proof. Obviously, the map Φ_A is linear for every $A \in V\left[\left[z^{\pm}\right]\right]$. For $f = \sum_{j \in \mathbb{Z}} f_j z^j$ we have

$$\mathrm{Res}_z A(z) f(z) = \sum_{j \in \mathbb{Z}} A_j f_{-j-1}, \tag{4.1}$$

which is well-defined, since for $f \in \mathbb{C}[z^{\pm}]$ only finitely many coefficients in the expansion of f may be non-zero. Let us now investigate the linear map $\Phi : V\left[\left[z^{\pm}\right]\right] \to \mathrm{Hom}\left(\mathbb{C}\left[z^{\pm}\right], V\right)$. If $\Phi_A = 0$ we have with eq. (4.1)

$$\sum_{j \in \mathbb{Z}} A_j f_{-j-1} = 0, \quad \text{forall} \quad f = \sum f_j z^j$$

implying $A = 0$ and we verified that the map Φ is injective. To show the surjectivity of Φ, we take an arbitrary element $\mu \in \mathrm{Hom}\left(\mathbb{C}\left[z^{\pm}\right], V\right)$ and define elements in the vector space V by $A_j := \mu(z^{-j-1})$. The formal distribution $A(z) := \sum_{j \in \mathbb{Z}} A_j z^j$ satisfies $\Phi(A)(z^{-j-1}) = \mu(z^{-j-1})$. Conclusively, $\Phi(A)$ and μ agree on a basis of $\mathbb{C}\left[z^{\pm}\right]$ and thus on all Laurent polynomials. □

For multiple variables, the ring of formal LAURENT series $C((z_1,\dots,z_n))$ is defined as the field of fractions of the ring $\mathbb{C}[[z_1,\dots,z_n]]$. Interestingly, some of the formal LAURENT series have no (or multiple) corresponding elements in the space of formal distributions. For example, we observe that the function $f(z-w)=(z-w)^{-1}\in\mathbb{C}((z,w))$ has two expansions in $\mathbb{C}\left[\left[z^{\pm},w^{\pm}\right]\right]$

$$f(z-w)=(z-w)=\begin{cases}\frac{1}{z}\sum_{n=0}^{\infty}z^{-n}w^{n} & \text{for } |z|>|w|,\\ -w\sum_{n=0}^{\infty}z^{n}w^{-n-1} & \text{for } |z|<|w|.\end{cases}$$

The difference between these two expansions is given by the formal delta distribution.

Definition 4.2. We define the *formal delta function* as the formal distribution

$$\delta(z-w):=\sum_{\substack{n,k\in\mathbb{Z}\\ n+k+1=0}}z^{n}w^{k}=\delta(w-z)\quad\in\mathbb{C}\left[\left[z^{\pm},w^{\pm}\right]\right].$$

Before we try to motivate this definition, we note that for a formal distribution $f=\sum_{j\in\mathbb{Z}}f_jz^j\in V\left[\left[z^{\pm}\right]\right]$ the product $f(z)\delta(z-w)$ is well-defined, because

$$\delta(z-w)f(z)=\sum_{j\in\mathbb{Z}}\sum_{n+k+1=0}f_jz^{j+n}w^{k}=\sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}f_jz^{j+n}w^{-n-1}=\sum_{k\in\mathbb{Z}}\left(\sum_{m\in\mathbb{Z}}f_{k+m+1}w^{m}\right)z^{k}\tag{4.2}$$

and should be regarded as an element of $\left(V\left[\left[w^{\pm}\right]\right]\right)\left[\left[z^{\pm}\right]\right]$. With eq. (4.1) and eq. (4.2), it follows that δ has indeed the properties of a delta function[1]

$$\Phi_\delta\left(f(z)\right)=\mathrm{Res}_z\delta(z-w)f(z)=f(w),$$
$$\Phi_\delta\left(f(w)\right)=\mathrm{Res}_w\delta(z-w)f(w)=f(z).$$

In the following, we will use the abbreviation $D_z^j:=\frac{1}{j!}\partial_z^j$ with $D_z^0=\mathbb{1}$. For a general rational function in two variables $F:\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ with poles only at $z=w=0$ and $|z|=|w|$, we have two LAURENT series expansion of F; one in the region $\{(z,w)\in\mathbb{C}^2\,|\,|z|>|w|\}$ and the other one in $\{(z,w)\in\mathbb{C}^2\,|\,|z|<|w|\}$. The former will be denoted by $\imath_{z,w}F$ and the latter by $\imath_{w,z}F$. As an example we could write the formal delta distribution as

$$\delta(z-w)=\imath_{z,w}\frac{1}{z-w}-\imath_{w,z}\frac{1}{z-w}.$$

Furthermore, the derivatives of the delta distribution can be expressed as the difference between the two expansions of $\frac{1}{(z-w)^{j+1}}$

$$\begin{aligned}\imath_{z,w}\frac{1}{(z-w)^{j+1}}&=\sum_{m=0}^{\infty}\binom{m}{j}z^{-m-1}w^{m-j}\\ \imath_{w,z}\frac{1}{(z-w)^{j+1}}&=-\sum_{m=-1}^{-\infty}\binom{-m}{j}z^{m-1}w^{-m-j}\end{aligned}\tag{4.3}$$

with

$$D_z^j\delta(z-w)=\imath_{z,w}\frac{1}{(z-w)^{j+1}}-\imath_{w,z}\frac{1}{(z-w)^{j+1}}.\tag{4.4}$$

[1] To be precise, one must regard $\delta(z-w)$ as an element of $\left(\mathbb{C}\left[\left[w^{\pm}\right]\right]\right)\left[\left[z^{\pm}\right]\right]$ resp. $\left(\mathbb{C}\left[\left[z^{\pm}\right]\right]\right)\left[\left[w^{\pm}\right]\right]$ for using Φ as above.

Lemma 4.2. *The formal delta function satisfies for all* $j, n \in \mathbb{N}_0$

$$(z-w)^n D_z^j \delta(z-w) = \begin{cases} 0 & \text{for } n > j, \\ D_z^{j-n}\delta(z-w) & \text{for } n \leq j. \end{cases}$$

Proof. First, we want to show that $(z-w)\delta(z-w) = 0$. Starting from eq. (4.2), we have for any formal distribution $f \in V[[z^{\pm}]]$

$$\delta(z-w)f(z) = \sum_{k\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}} f_{k+m+1} w^m \right) z^k = \sum_{k\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}} f_{k+m+1} z^m \right) w^k = \delta(z-w) f(w).$$

Taking $f(z) = z$ yields the desired relation. Furthermore, we can derive an recurrence relation for the derivatives acting on the formal delta distribution [2]

$$\begin{aligned}
(z-w)\partial_z^{j+1}\delta(z-w) &= (z-w)\sum_{m\in\mathbb{Z}} m(m-1)\dots(m-j) z^{-m-j} w^{m-1} \\
&= \sum_{m\in\mathbb{Z}} m(m-1)\dots(m-j)\left(z^{-m-j+1}w^{m-1} - z^{-m-j}w^m\right) \\
&= \sum_{m\in\mathbb{Z}} \big((m+1)m\dots(m-j+1) - m(m-1)\dots(m-j)\big) z^{-m-j} w^m \\
&= (j+1)\sum_{m\in\mathbb{Z}} m(m-1)\dots(m-j) z^{-m-j} w^m \\
&= (j+1)\partial_z^j \delta(z-w).
\end{aligned}$$

This shows that $(z-w)D_z^{j+1}\delta(z-w) = D_z^j\delta(z-w)$ and finally proves the lemma. □

Corollary 4.1. *Let* $f \in V[[z^{\pm}, w^{\pm}]]$ *with* $(z-w)^N f(z,w) = 0$ *for some* $N \in \mathbb{N}$*. Then, we can expand* f *in terms of derivatives of the formal delta function via*

$$f(z,w) = \sum_{j=0}^{N-1} c^j(w)\, D_w^j \delta(z-w),$$

where the coefficients c^j *for* $j \in \{0, \dots N-1\}$ *are given by*

$$c^j(w) = \mathrm{Res}_z (z-w)^j f(z,w) \in V[[w^{\pm}]].$$

Proof. We will prove this corollary for $f = \sum_{m,n\in\mathbb{Z}} f_{mn} z^m w^n$ by induction. If $(z-w)^N f(z,w)$ for $N = 1$ vanishes, we have

$$0 = \sum_{m,n\in\mathbb{Z}} \left(f_{mn} z^{m+1} w^n - f_{mn} z^m w^{n+1} \right) = \sum_{m,n\in\mathbb{Z}} \left(f_{m,n+1} - f_{m+1,n} \right) z^m w^n,$$

and thus $f_{m+1,n} = f_{m,n+1}$ for all $m, n \in \mathbb{Z}$. This implies that we can write f as

$$f(z,w) = \sum_{m,k\in\mathbb{Z}} f_{m,k-m-1} z^m w^{k-m-1} = \sum_{m,k\in\mathbb{Z}} f_{1,k} w^k z^m w^{-m-1} = c^0(w)\delta(z-w),$$

[2] We used for the fourth equality

$$\begin{aligned}
(m+1)m\dots(m-j+1) - m(m-1)\dots(m-j) &= \frac{(m+1)!}{(m-j)!} - \frac{m!}{(m-j-1)!} = \frac{(m+1)! - m!(m-j)}{(m-j)!} \\
&= \frac{m!\big(m+1-(m-j)\big)}{(m-j)!} = (j+1)\frac{m!}{(m-j)!}.
\end{aligned}$$

where the coefficient $c^0(w) = \sum_{k\in\mathbb{Z}} f_{1,k} w^k = \mathrm{Res}_z f(z,w)$ is in accordance with the corollary. Now let us assume that the expansion from the corollary is possible for $N = n$ and that $(z-w)^{n+1} f(z,w) = (z-w)^n (z-w) f(z,w) = 0$. Then, we can expand $(z-w)f(z,w)$ with coefficients $d^j \in V[[w^{\pm}]]$ as

$$(z-w)f(z,w) = \sum_{j=0}^{n-1} d^j(w)\, D_w^j \delta(z-w). \tag{4.5}$$

The condition that $(z-w)^{n+1}$ annihilates f furthermore implies that

$$\partial_z\big((z-w)^{n+1} f\big) = (n+1)(z-w)^n f + (z-w)^{n+1}\partial_z f = (z-w)^n \underbrace{\big((z-w)\partial_z f + (n+1)f\big)}_{\stackrel{!}{=}0}, \tag{4.6}$$

yielding another expansion

$$\big((z-w)\partial_z f + (n+1)f\big) = \sum_{i=0}^{n-1} e^j(w)\, D_w^j \delta(z-w). \tag{4.7}$$

Differentiating eq. (4.5) with respect to z leaves us with the expression

$$f + (z-w)\partial_z f = -\sum_{j=0}^{n-1} d^j(w)(j+1) D^{j+1}\delta(z-w). \tag{4.8}$$

According to eq. (4.6), the difference between eq. (4.7) and eq. (4.8) is given by nf. Conclusively, we receive an expression for f as[3]

$$\begin{aligned} f(z,w) &= \frac{1}{n}\sum_{j=0}^{n} e^j(w)\, D_w^j\delta(z-w) + \frac{1}{n}\sum_{j=0}^{n} j d^{j-1}(w)\, D_z^j \delta(z-w) \\ &= \sum_{j=0}^{n} c^j(w)\, D_w^j \delta(z-w), \\ \text{with } c^j(w) &:= \frac{1}{2} e^j(w) + \frac{j}{n} d^{j-1}(w), \text{ for } j \in \{0,\dots,n\}. \end{aligned}$$

Finally, we want to show that if f can be expanded as stated in the corollary, the c^js are uniquely determined and given by the formula in the corollary. We derive with lemma 4.2

$$\begin{aligned} \mathrm{Res}_z (z-w)^k f(z,w) &= \mathrm{Res}_z \sum_{j=n}^{N-1} c^j(w)\, D_w^{j-k}\delta(z-w) \\ &= \mathrm{Res}_z \sum_{j=n}^{N-1} c^j(w) \sum_{l\in\mathbb{Z}} l(l-1)\dots(l-(j-k)+1) z^{-l-1} w^l \\ &= \sum_{j=n}^{N-1} c^j(w)\delta_{k,j} = c^k(w). \end{aligned}$$

This ends the proof, since the residue of a formal distribution is unique. □

For the upcoming discussion, we want V to be an associative $\mathbb{C}$-algebra. As usual, we define the *commutator* on V by $[X,Y] = XY - YX$ for $X, Y \in V$. Additionally, we introduce a new notation for the coefficients of a formal distribution $A \in V[[z^{\pm}]]$ as $A_{(n)} := A_{-n-1}$, yielding for a distribution $A = \sum_{n\in\mathbb{Z}} A_n z^n$ the expansion $A = \sum_{n\in\mathbb{Z}} A_{(n)} z^{-n-1}$. We denote the part of A with non-negative (negative) exponents as

$$A(z)_+ := \sum_{n=0}^{\infty} A_{(n)} z^{-n-1} \quad \text{resp.} \quad A(z)_- := \sum_{n=-1}^{-\infty} A_{(n)} z^{-n-1},$$

with the property $(\partial A(z))_\pm = \partial(A(z)_\pm)$.

[3]Here, both $e^n(w) := 0$ and $d^{-1}(w) := 0$ are defined properly.

Definition 4.3. We call two distributions $A, B \in V[[z^{\pm}]]$ *local* with respect to each other if

$$(z-w)^N [A(z), B(w)] = 0$$

for some $N \in \mathbb{N}$.

Definition 4.4. Let $A, B \in V[[z^{\pm}]]$. The *normally ordered product* of A and B is given by

$$:A(z)B(w): \;:= A(z)_+ B(w) + B(w)A(z)_- \quad \in V[[z^{\pm}, w^{\pm}]].$$

Corollary 4.2. *Two formal distributions $A, B \in V[[z^{\pm}]]$ are local with $N \in \mathbb{N}$, i.e. $(z-w)^N [A(z), B(w)] = 0$, if and only if there exist formal distributions $C^j \in V[[z^{\pm}]]$ with*

$$A(z)B(w) = \sum_{j=0}^{N-1} \iota_{z,w} \frac{1}{(z-w)^{j+1}} C^j(w) + \; :A(z)B(w):,$$

$$B(w)A(z) = \sum_{j=0}^{N-1} \iota_{w,z} \frac{1}{(z-w)^{j+1}} C^j(w) + \; :A(z)B(w):.$$

Proof.

"⇒" If $(z-w)^N$ annihilates the distribution $[A(z), B(w)]$, corollary 4.1 implies that there exist $C^j \in V[[w^{\pm}]]$, such that the commutator can be written as

$$[A(z), B(w)] = \sum_{j=0}^{N-1} C^j(w) D^j \delta(z-w).$$

This is an equality between two formal series on all of $\mathbb{C}^2$. With eq. (4.4), we can split this equation into the two contributions from $|z| > |w|$ and $|z| < |w|$ as

$$[A(z)_-, B(w)] = \sum_{j=0}^{N-1} \iota_{z,w} \frac{C^j(w)}{(z-w)^{j+1}}$$

$$[A(z)_+, B(w)] = -\sum_{j=0}^{N-1} \iota_{w,z} \frac{C^j(w)}{(z-w)^{j+1}}.$$

With the definition of the normally ordered product we now find

$$A(z)B(w) = [A(z)_-, B(w)] + \; :A(z)B(w): = \sum_{j=0}^{N-1} \iota_{z,w} \frac{C^j(w)}{(z-w)^{j+1}} + \; :A(z)B(w):$$

$$B(w)A(z) = -[A(z)_+, B(w)] + \; :A(z)B(w): = \sum_{j=0}^{N-1} \iota_{w,z} \frac{C^j(w)}{(z-w)^{j+1}} + \; :A(z)B(w):.$$

"⇐" If the products $A(z)B(w)$ and $B(w)A(z)$ are given as stated in the theorem, we immediately have with lemma 4.2

$$\begin{aligned}(z-w)^N [A(z), B(w)] &= \sum_{j=0}^{N-1} \iota_{z,w} C^j(w)(z-w)^{N-j-1} + \; :A(z)B(w): \\ &\quad - \sum_{j=0}^{N-1} \iota_{w,z} C^j(w)(z-w)^{N-j-1} - \; :A(z)B(w): \\ &= 0,\end{aligned}$$

where the last equality holds, because the two expansions $\iota_{z,w}(z-w)^{N-j-1}$ and $\iota_{w,z}(z-w)^{N-j-1}$ are identical for $j \in \{0,\dots N-1\}$.

□

In the region where $|z| > |w|$, we denote the singular part of the product $A(z)B(w)$, i.e. the difference between $A(z)B(w)$ and $:A(z)B(w):$, by writing a tilde instead of an equal sign. In the specific expansion from above, this would imply

$$A(z)B(w) \sim \sum_{j=0}^{N-1} \frac{C^j(w)}{(z-w)^{j+1}}. \tag{4.9}$$

To introduce the concept of a field, we let the $\mathbb{C}$-algebra of coefficients (formerly denoted by V) be the space of linear operators over some complex vector space W.

Definition 4.5. A *field* is a formal distribution

$$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \quad \text{with} \quad a_{(n)} \in \mathrm{End}(W),$$

where for every $v \in W$ we have that $a(z)(v)$ is a formal Laurent series, i.e.there exists $N \in \mathbb{N}$ with

$$a_{(n)}(v) = 0 \quad \forall n \geq N.$$

The vector space of fields over W will be denoted by $\mathscr{F}(W)$. We want to give an example of such a field by taking V as a representation of the Heisenberg algebra H generated by the elements (cf. section 2.3) a_n, Z with $n \in \mathbb{Z}$, where

$$[a_n, a_m] = m\delta_{m+n} Z \quad \text{and} \quad [a_n, Z] = 0 \qquad \forall n, m \in \mathbb{Z}.$$

We use the space of polynomials $S = \mathbb{C}[T_1, T_2, \dots]$ as the representation space, where we define the representation for $n \in \mathbb{N}$ as [4]

$$\begin{aligned} a_n &= \frac{\partial}{\partial_n}, \\ a_0 &= 0, \\ a_{-n} &= nT_n, \\ Z &= \mathbb{1}_S. \end{aligned}$$

We define the formal distribution $A(z) := \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$. For any element $P \in S$ with T_m as the indeterminate with the highest index, we have $a_{(n)}P = 0$ for all $n \geq m+1$, making $A(z)$ a field. The commutator of this field with itself satisfies

$$\begin{aligned} [A(z), A(w)] &= \sum_{m,n\in\mathbb{Z}} [a_m, a_n] z^{-m-1} w^{-n-1} \\ &= \sum_{m\in\mathbb{Z}} m z^{-m-1} w^{m-1} Z \\ &= \partial_w \delta(z-w) Z. \end{aligned}$$

With lemma 4.2 this implies that $A(z)$ is local to itself with $N = 2$. Furthermore, corollary 4.2 implies that the singular part of the product of $A(z)$ with itself is given by

$$A(z)A(w) \sim \frac{Z}{(z-w)^2} = \frac{1}{(z-w)^2}. \tag{4.10}$$

[4] We again omit writing the symbol for the representation.

Let W be a vector space that can be decomposed into finite-dimensional subspaces W_n as the direct sum $W = \bigoplus_{n\in\mathbb{Z}} W_n$, where $W_n = \{0\}$ for $n < 0$. A homomorphism $T \in \mathrm{End}(W)$ is called *homogeneous of degree* g if $T(W_n) \subset W_{n+g}$.

Definition 4.6. A field $a = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ is called *homogeneous of conformal weight* h, if every endomorphism $a_{(n)}$ is homogeneous of degree $h - n - 1$.

Lemma 4.3. *If the fields $A(z), B(z)$ are homogeneous of conformal weight h resp. h', then their normally ordered product $:A(z)B(z):$ is homogeneous of conformal weight $h + h'$.*

Proof. We begin by expanding the normally ordered product, to identify the coefficient $:A(z)B(z):_{(k)}$

$$\begin{aligned} :A(z)B(z): &= \sum_{m\in\mathbb{Z}} \left(\sum_{n=-1}^{-\infty} A_{(n)} B_{(n)} z^{-(m+n)-2} + \sum_{n=0}^{\infty} B_{(m)} A_{(n)} z^{-(m+n)-2} \right) \\ &= \sum_{k\in\mathbb{Z}} \underbrace{\left(\sum_{n=-1}^{-\infty} A_{(n)} B_{(k-n-1)} + \sum_{n=0}^{\infty} B_{(k-n-1)} A_{(n)} \right)}_{:A(z)B(z):_{(k)}} z^{-k-1}. \end{aligned}$$

We know that $A_{(m)}$ is homogeneous of degree $h - m - 1$ and that $B_{(m)}$ is homogeneous of degree $h' - m - 1$, i.e.

$$\begin{aligned} A_{(m)} V_j &\subset V_{j+h-m-1}, \\ B_{(m)} V_j &\subset V_{j+h'-m-1}. \end{aligned}$$

Combining these two equations yields

$$\begin{aligned} A_{(n)} B_{(k-n-1)} V_j &\subset V_{j+h+h'-k-1}, \\ B_{(k-n-1)} A_{(n)} V_j &\subset V_{j+h+h'-k-1}. \end{aligned}$$

Thus, $:A(z)B(z):_{(k)}$ is homogeneous of degree $h + h' - k - 1$ and we have proven the statement. □

Lemma 4.4. *If the field $A(z)$ is homogeneous of conformal weight h, then $\partial A(z)$ is homogeneous with conformal weight $h + 1$.*

Proof. Let $A(z) = \sum_{n\in\mathbb{Z}} A_{(n)} z^{-n-1}$ with $A_{(k)} V_j \subset V_{j+h-k-1}$. Then the derivative of $A(z)$ is given by $\partial A(z) = \sum_{n\in\mathbb{Z}} (-n) A_{(n-1)} z^{-n-1}$, implying that for its coefficients we have

$$(\partial A(z))_{(k)} V_j \propto A_{(k-1)} V_j \subset V_{j+(h+1)-k-1}.$$

□

The representation space $S = \mathbb{C}[T_1, T_2, \ldots]$ of the Heisenberg algebra has a natural decomposition into the subspaces of constant degree of the polynomial $\deg\left(T_{n_1} \ldots T_{n_k}\right) \equiv \sum_{j=1}^{k} n_j$. Additionally, $a_{(n)}$ is homogeneous of degree $-n$ implying that $a(z)$ is homogeneous of conformal weight 1.

4.2 Introduction to Vertex Algebras

Definition 4.7. A *vertex algebra* consists of a vector space V with a vacuum vector Ω, an infinitesimal translation operator $T \in \mathrm{End}(V)$, and a vertex operator $Y \in \mathrm{Hom}(V, \mathscr{F}(V))$

$$a \longmapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}.$$

Furthermore, we require a vertex algebra to satisfy the following three axioms:

Translation Covariance: $[T, Y(a,z)] = \partial Y(a,z)$,

Locality: $\forall a,b \in V\ \exists N \in \mathbb{N}: \quad (z-w)^N [Y(a,z), Y(b,w)] = 0$,

Vacuum: $T\Omega = 0, \quad Y(\Omega,z) = \mathbb{1}_V, \quad \text{and} \quad Y(a,z)\Omega|_{z=0} = a.$

Before we can prove an important theorem about the existence of a vertex algebra, we need the following lemma.

Lemma 4.5 (Dong's lemma). *Let V be a vector space and $a(z), b(z)c(z) \in \mathscr{F}(V)$ pairwise local fields. Then $:a(z)b(z):$ is local with respect to $c(w)$.*

We will follow the proof published in [1] for $n = -1$.

Proof. With eq. (4.3) we get the formulae

$$\begin{aligned} \mathrm{Res}_{z=w} a(z)\, \iota_{z,w} \frac{1}{z-w} &= a(w)_+ \\ \mathrm{Res}_{z=w} a(z)\, \iota_{w,z} \frac{1}{z-w} &= -a(w)_- \end{aligned}$$

yielding an alternative formulation of the normally ordered product between two formal distributions in the same indeterminate

$$:a(w)b(w): = \mathrm{Res}_{z=w}\left(\iota_{z,w}\frac{1}{z-w}a(z)b(w) - \iota_{w,z}\frac{1}{z-w}b(w)a(z)\right).$$

In order to prove the theorem, we need to show that there exists $N \in \mathbb{N}$ such that

$$(z_2 - z_3)^N A(z_1,z_2,z_3) = (z_2 - z_3)^N B(z_1,z_2,z_3), \tag{4.11}$$

where the two fields A and B are given by

$$\begin{aligned} A(z) &= \iota_{z_1,z_2} \frac{a(z_1)b(z_2)}{z_1 - z_2} c(z_3) - \iota_{z_2,z_1} \frac{b(z_2)a(z_1)}{z_1 - z_2} c(z_3), \\ B(z) &= \iota_{z_1,z_2} c(z_3) \frac{a(z_1)b(z_2)}{z_1 - z_2} - \iota_{z_2,z_1} c(z_3) \frac{b(z_2)a(z_1)}{z_1 - z_2}. \end{aligned}$$

The residue of eq. (4.11) at $z_1 = z_2$ then yields the locality between $:a(z_1)b(z_2):$ and $c(z_3)$. Let $r \in \mathbb{N}$, such that

$$\begin{aligned} (z_1 - z_2)^r [a(z_1), b(z_2)] &= 0, \\ (z_2 - z_3)^r [b(z_2), c(z_3)] &= 0, \\ (z_3 - z_1)^r [c(z_3), a(z_1)] &= 0. \end{aligned}$$

If we set $M = 3r$, the left hand side of eq. (4.11) becomes

$$(z_2 - z_3)^r \sum_{i=0}^{2r} \binom{2r}{i} (z_2 - z_1)^{2r-i} (z_1 - z_3)^i A,$$

where we used the binomial theorem $(x+y)^m = \sum_{j=0}^m \binom{m}{j} x^{m-j} y^j$ for $x = z_2 - z_1$ and $y = z_1 - z_3$. The terms in this series with $2r - i - 1 \geq r \Leftrightarrow i \leq r - 1$ are proportional to

$$\begin{aligned}(z_1 - z_2)^{2r-i}\iota_{z_1,z_2}\frac{1}{z_1 - z_2}a(z_1)b(z_2) - (z_1 - z_2)^{2r-i}\iota_{z_2,z_1}\frac{1}{z_1 - z_2}b(z_2)a(z_1)\\ = (z_1 - z_2)^{r'}\left[a(z_1), b(z_2)\right]\\ = 0,\end{aligned}$$

with $r' \geq r$ by the condition on i. Performing the analogous calculation for the right hand side, eq. (4.11) reduces to

$$(z_2 - z_3)^r \sum_{i=r+1}^{2r} \binom{2r}{i} (z_2 - z_1)^{2r-i} (z_1 - z_3)^i A \stackrel{?}{=} (z_2 - z_3)^r \sum_{i=r+1}^{2r} \binom{2r}{i} (z_2 - z_1)^{2r-i} (z_1 - z_3)^i B.$$

Because of the locality of $b(z_2)$ and $c(z_3)$ at order r, this relation is fulfilled and we conclude that eq. (4.11) holds for $N \geq 3r$. □

Theorem 4.1. *Let V be a vector space with an element $\Omega \in V$, T an endomorphism $T \in \mathrm{End}(V)$. Furthermore, let $I \subset V$ with $(\Phi_a(z))_{i\in I}$ be a set of fields given by*

$$\Phi_a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \in \mathrm{End}(V)\left[\left[z^{\pm}\right]\right].$$

If these objects satisfy for all $a, b \in I$

i) $[T, \Phi_a(z)] = \partial\Phi_a(z)$,

ii) $T\Omega = 0$ *and* $\Phi_a(z)\Omega|_{z=0} = a$,

iii) $\exists N \in \mathbb{N}: (z - w)^N \left[\Phi_a(z), \Phi_b(w)\right] = 0$,

iv) $\left\{a^1_{(-k_1)} \dots a^n_{(-k_n)}\Omega \,\middle|\, a^j \in I,\ k_j \in \mathbb{N}\right\} \cup \{\Omega\}$ *is a basis of V,*

then this yields a vertex algebra with translation operator T, vacuum Ω, and vertex operator

$$Y\left(a^1_{(-k_1)} \dots a^n_{(-k_n)}\Omega, z\right) = \ :D^{k_1-1}\Phi_{a^1}(z) \dots D^{k_n-1}\Phi_{a^n}(z):.$$

This furthermore implies $Y(a,z) = \Phi_a(z)$ for $a \in I$.

Proof. Let us start with the last claim, that $Y(a,z) = \Phi_a(z)$. The second point implies $a_{(-1)}\Omega = a$, yielding

$$Y(a,z) = Y(a_{(-1)}\Omega, z) \equiv\ :D^0\Phi_a(z): = \Phi_a(z).$$

Because of point iv), the vertex operator is well-defined on all of V. We have to verify, that the axioms of a vertex algebra are fulfilled under these circumstances. Point i) yields that $\mathrm{ad}_T|_I = \partial|_I$. It remains to show that a field of the form $:D^{k_1-1}\Phi_{a^1}(z)\dots D^{k_n-1}\Phi_{a^n}(z):$ behaves in the same way under the action of T and ∂. The derivative acts in a natural way on a normally ordered product of two fields $a(z), b(z)$, since $\partial\left(a(z)_\pm\right) = \left(\partial a(z)\right)_\pm$

$$\begin{aligned}\partial :a(z)b(z): &= \partial\left(a(z)_+ b(z) + b(z)a(z)_-\right)\\ &= \partial a(z)_+ b(z) + a(z)_+\partial b(z) + \partial b(z)a(z)_- + b(z)\partial a(z)_-\\ &= \ :\partial a(z)b(z): + \ :a(z)\partial b(z):.\end{aligned}$$

On the other hand, it generally holds that

$$\begin{aligned}\mathrm{ad}_T\left(a(z)b(z)\right) &= \left[T, a(z)b(z)\right]\\ &= \left[T, a(z)\right]b(z) + a(z)\left[T, b(z)\right]\\ &= \mathrm{ad}_T a(z)b(z) + a(z)\mathrm{ad}_T b(z)\end{aligned}$$

for any endomorphism $T \in \mathrm{End}(V)$. Obviously, the partial derivative commutes with the operator $D^j = \frac{1}{j!}\partial^j$ for all $j \in \mathbb{N}$. The same holds for ad_T, since T resp. ad_T is independent of z.

Concerning the locality axiom, Dong's lemma (lemma 4.5) implies that all fields of the form $:D^{k_1-1}\Phi^1_a(z)\dots D^{k_n}\Phi_{a^n}(z):$ are local if the fields $\Phi_{a^j}(z)$ are pairwise local. This is a consequence of the locality of $\partial A(z)$ and $B(z)$ given that $A(z)$ and $B(z)$ are local. This simply follows from

$$(z-w)^{N+1}\left[\partial A(z), B(w)\right] = \partial_z\left((z-w)^{N+1}\left[A(z), B(w)\right]\right) - (N+1)(z-w)^N\left[A(z), B(w)\right] = 0,$$

if we assume that the field are local with exponent N. Finally, point ii) ensures that the vacuum axiom holds. □

Going back to the case of the Heisenberg algebra (cf. section 4.1), we can use theorem 4.1 to construct the Heisenberg vertex algebra. We will use the same representation of H as before. The set we mentioned in point iv) – in this case the set of all monomials – is a basis of the space S. In analogy to theorem 4.1, we denote $\Phi_a(z) = A(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$, where a is defined via point ii) of the theorem

$$a := \Phi_a(z)\Omega\Big|_{z=0} = a_{-1}\Omega = T_1.$$

Thus, we write $I = \{T_1\}$. Point iii), i.e. the locality of $A(z)$ with itself, was already shown in eq. (4.10). It remains to construct a suitable translation operator T. In order for ad_T to act as a derivation on $A(z)$, we have

$$[T, a_n] = -na_{n-1} \quad \text{and} \quad T\Omega = 0.$$

This relation defines the action of ad_T on the basis elements of S and thus yields a unique endomorphism. We can show

$$\begin{aligned}\left[\sum_{m\in N} a_{-m-1}a_m, a_n\right] &= \sum_{m\in\mathbb{N}} [a_{-m-1}a_m, a_n] \\ &= \sum_{m\in\mathbb{N}} \left(a_{-m-1}[a_m, a_n] + [a_{-m-1}, a_n]\, a_m\right) \\ &= \sum_{m\in\mathbb{N}} \left(ma_{-m-1}\delta_{m+n} + (-m-1)a_m\delta_{-m-1+n}\right) \\ &= -na_{n-1}\end{aligned} \tag{4.12}$$

and conclude that we should identify $T = \sum_{n\in\mathbb{N}} a_{-n-1}a_n$. Now all the conditions of theorem 4.1 are fulfilled and we conclude that

$$Y\left(T_{k_1}\dots T_{k_n}, z\right) = \;:D^{k_1-1}A(z)\dots D^{k_n-1}A(z):, \quad \text{with} \quad A(z) = Y(T_1, z) \tag{4.13}$$

induces a vertex algebra structure on S.

Definition 4.8. A *graded vertex algebra* is a vertex algebra on a vector space V with

$$V = \bigoplus_{n=0}^{\infty} V_n, \quad \dim(V_n) < \infty,$$

that satisfies

i) $V_0 = \mathbb{C}\Omega$,

ii) T is homogeneous of degree 1,

iii) $Y(a,z)$ is homogeneous of conformal weight m for $a \in V_m$.

With the same decomposition we used before, $S_n := \{P \in S \mid \deg(P) = n\}$, the Heisenberg vertex algebra trivially fulfils conditions i) and ii) of a graded vertex algebra. The last condition can be shown using lemma 4.3 and lemma 4.4, making the Heisenberg vertex algebra a graded vertex algebra.

4.3 Conformal Vertex Algebra

Another interesting example of applying theorem 4.1 is the construction of the VIRASORO vertex algebra. In definition 4.6 we defined the conformal weight of a field $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ as h, where every endomorphism $a_{(n)}$ is homogeneous of degree $h-n-1$. In the case of the HEISENBERG vertex algebra, the representation space was graded in a way, that made the coefficients a_n homogeneous of degree $-n$. Before we start with the construction of the VIRASORO vertex algebra, we want to give an insight into a different approach to the conformal weights. Let H be the HAMILTONian of the system, i.e. a diagonalisable derivation on the $\mathbb{C}$-algebra $\mathsf{End}(V)$. We say a field $a(z) \in \mathscr{F}(V)$ is of conformal weight $\Delta \in \mathbb{C}$, if

$$(H - \Delta - z\partial_z)\, a(z) = 0$$

and refer to it as

$$a(z) = \sum_{n\in\mathbb{Z}-\Delta} a_n z^{-n-\Delta}. \tag{4.14}$$

This definition will especially become important, when we consider superconformal vertex algebras in section 5.3. For reasons that will become apparent later, we need to define the VIRASORO field as a field of conformal weight 2

$$L(z) \coloneqq \sum_{n\in\mathbb{Z}} L_n z^{-n-2}.$$

We define a representation on the vector space V_c with the basis

$$\left\{ v_{n_1\dots n_k} \;\middle|\; 1 < n_k \leq \dots n_1,\ n_j \in \mathbb{N},\ k \in \mathbb{N} \right\} \cup \{\Omega\}.$$

This space is very similar to the VERMA module $M(c,0)$ (cf. definition 3.4), with the distinction that $n_j > 1$ instead of $n_j > 0$. To understand why this restriction is necessary, we note that

$$[L_{-1}, L(z)] = \sum_{n\in\mathbb{Z}} [L_{-1}, L_n]\, z^{-n-2} = \sum_{n\in\mathbb{Z}} (-1-n) L_{n-1} z^{-n-2} = \partial L(z).$$

Thus, we would like to identify the infinitesimal translation vector T with L_{-1}, implying that $L_{-1}\Omega \overset{!}{=} 0$. The representation is defined similar to the construction of the VERMA module in the proof of lemma 3.1 for $h = 0$

$$\begin{aligned} Z &\coloneqq c\mathbb{1}_{V_c}, \\ L_0 v_{n_1\dots n_k} &\coloneqq \left(\sum_{i=1}^{k} n_i\right) v_{n_1\dots n_k}, \\ L_n\Omega &\coloneqq \begin{cases} 0 & \text{for } n \geq -1 \\ v_{-n} & \text{for } n < -1 \end{cases}, \\ L_{-n} v_{n_1\dots n_k} &\coloneqq v_{n n_1\dots n_k} \quad \text{for} \quad n \geq n_1, \end{aligned} \tag{4.15}$$

where the remaining relations are again defined by the commutation relations of the L_ns. With this definition it is clear, that $L(z)$ is indeed a field, since for every $v_{n_1\dots n_k} \in V$ there exists a $N \in \mathbb{N}$ with $L_n v_{n_1\dots n_k} = 0$ for $n \geq N$. To verify that it is local to itself, we first calculate the commutator of the field

$$\begin{aligned} [L(z), L(w)] &= \sum_{m,n\in\mathbb{Z}} [L_m, L_n]\, z^{-m-2} w^{-n-2} \\ &= \sum_{m,n\in\mathbb{Z}} (m-n) L_{m+n} z^{-m-2} w^{-n-2} + \sum_{m\in\mathbb{Z}} \frac{m}{12}(m^2-1) z^{-m-2} w^{m-2} Z \\ &= \sum_{k,m\in\mathbb{Z}} (2m-k) L_k z^{-m-2} w^{-k-2} w^m + \sum_{m\in\mathbb{Z}} \frac{Z}{12}(m+1)m(m-1) z^{-m-2} w^{m-2} \\ &= 2\sum_{k\in\mathbb{Z}} L_k w^{-k-2} \sum_{m\in\mathbb{Z}} (m+1) z^{-m-2} w^m - \sum_{k\in\mathbb{Z}} (k+2) L_k z^{-k-3} \sum_{m\in\mathbb{Z}} z^{-m-2} w^{m+1} + \frac{Z}{12}\partial_w^3 \delta(z-w) \\ &= 2L(w)\partial_w \delta(z-w) + \partial_w L(w)\delta(z-w) + \frac{Z}{12}\partial_w^3 \delta(z-w). \end{aligned} \tag{4.16}$$

With corollary 4.1 and corollary 4.2, we find the operator product expansion (cf. eq. (4.9))

$$L(z)L(w) \sim \frac{c}{2}\frac{1}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w}. \tag{4.17}$$

In order to achieve a vertex algebra structure on V_c with theorem 4.1, we define

$$a := L(z)\Omega\Big|_{z=0} = L_{-2}\Omega = v_2.$$

We summarise our findings in the following corollary.

Corollary 4.3. *The field $\Phi_a(z) := L(z)$ with $a = v_2$ combined with the vacuum vector $\Omega \in V_c$ and the infinitesimal translation vector $T := L_{-1}$ yield a vertex algebra structure on the space V_c.*

Proof. See the discussion above and theorem 4.1. □

Definition 4.9. A field with an operator product expansion of the form eq. (4.17) is called a *Virasoro field with central charge c*.

Definition 4.10. A vector $v \in V$ with the property that

$$Y(v,z) = \sum_{n\in\mathbb{Z}} v_{(n)} z^{-n-1} \equiv \sum_{n\in\mathbb{Z}} L_n^v z^{-n-2}$$

is a Virasoro field with central charge c, $T = L_{-1}^v$, and L_0^v is diagonalisable, is called a *conformal vector with central charge c*.

Definition 4.11. A *conformal vertex algebra of rank c* is a vertex algebra with a conformal vector with central charge c.

The field $Y(v,a)$, where v is a conformal vector, is called the *energy-momentum-tensor* of the vertex algebra. Obviously, the Virasoro vertex algebras V_c are conformal vertex algebras with conformal vectors $v_2 = L_{-2}\Omega$. We will take a look at a more interesting example of a conformal vertex algebra in the following.

Instead of just having one conformal vector, the Heisenberg vertex algebra $S = \mathbb{C}[T_1, T_2, \ldots]$ has a one-parameter family of conformal vectors. We postpone the proof of this statement until the end of the next section.

4.4 Operator Product Expansion

Lemma 4.6. *Let V be a vector space with $g \in V$ and $S \in \mathrm{End}(V)$. Then, the initial value problem in $V[[z]]$*

$$\partial f(z) = S f(z), \quad f(0) = g$$

has the unique solution

$$f(z) = \sum_{n\in\mathbb{N}} \left(\frac{S^n}{n!} g\right) z^n =: e^{Sz} g \quad \in V[[z]].$$

Proof. Let us assume that the solution of the initial value problem is given by $f(z) = \sum_{n\in\mathbb{N}} f_n z^n \in V[[z]]$. With the definition of the derivative of a formal distribution, we have

$$\partial f(z) = \sum_{n\in\mathbb{N}} (n+1) f_{n+1} z^n \stackrel{!}{=} \sum_{n\in\mathbb{N}} S f_n z^n,$$

yielding the condition $(n+1)f_{n+1} = Sf_n$ for $n \geq 0$. The condition $f(0) = g$ yields $f_0 = g$ and thus

$$f_n = \frac{S^n}{n!} f_0 = \frac{S^n}{n!} g.$$

□

Corollary 4.4. *Let V be a vertex algebra. Then, for every element $a \in V$ we have*

$$Y(a,z)\Omega = e^{zT} a.$$

Furthermore, in the space $\big(\mathrm{End}(V)\,[[z^{\pm}]]\big)[[w]]$ it holds that

$$e^{wT} Y(a,z) e^{-wT} = Y(a,z+w).$$

Proof. Let us start with the first equality. The translation covariance axiom combined with the vacuum axiom yield for any element $a \in V$

$$\partial Y(a,z)\Omega = [T, Y(a,z)]\,\Omega = TY(a,z)\Omega - Y(a,z)T\Omega = TY(a,z)\Omega.$$

Therefore, we can apply lemma 4.6 on the vector $Y(a,z)\Omega \in V$ with the initial condition $Y(a,z)\Omega|_{z=0} = a$, yielding the desired equality. For the second relation we will show that both sides satisfy one and the same initial value problem $\partial_w f = (\mathrm{ad}_T) f$ and use the uniqueness of such a solution (cf. lemma 4.6). We calculate

$$\partial_w \left(e^{wT} Y(a,z) e^{-wT}\right) = Te^{wT} Y(a,z) e^{-wT} + e^{wT} Y(a,z) e^{-wT}(-T) = \mathrm{ad}_T \left(e^{wT} Y(a,z) e^{-wT}\right)$$

and

$$\partial_w Y(a,z+w) = \partial_{w+z} Y(a,z+w) = [T, Y(a,z+w)]\,.$$

Since both elements have the same value for $w = 0$, that is $Y(a,z)$, the statement is proven. □

Corollary 4.5. *Let V be a vertex algebra. If a field $f \in \mathrm{End}(V)\,[[z^{\pm}]]$ is local with respect to $Y(a,z)$ for all $a \in V$ and satisfies*

$$f(z)\Omega = e^{zT} b$$

for some $b \in V$, then $f(z) = Y(b,z)$.

Proof. By assumption, for every $a \in V$ there exists an $N \in \mathbb{N}$, such that

$$\begin{aligned}
(z-w)^N f(z) e^{wT} a &= (z-w)^N f(z) Y(a,w)\Omega \\
&= (z-w)^N Y(a,w) f(z)\Omega \\
&= (z-w)^N Y(a,w) e^{zT} b \\
&= (z-w)^N Y(a,w) Y(b,z)\Omega \\
&= (z-w)^N Y(b,z) Y(a,w)\Omega \\
&= (z-w)^N Y(b,z) e^{wT} a,
\end{aligned}$$

where we have used corollary 4.4 for the last equality and the commutation of all fields $Y(c,z)$ for large enough N. Taking $w = 0$, we conclude that $f(z)a = Y(b,z)a$. Since a was chosen arbitrarily, this proves the statement. □

Corollary 4.6. *Let V be a vertex algebra. Then, for every element $a \in V$ we have*

$$Y(Ta, z) = \partial Y(a, z).$$

Proof. Let $Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} = \sum_{n\in\mathbb{Z}} a_{(-n-1)} z^n$. Using the translation covariance, we have the expansion

$$\partial Y(a,z)\Omega = TY(a,z)\Omega = \sum_{n\in\mathbb{Z}} Ta_{(-n-1)}\Omega z^n.$$

With the vacuum axiom for $Y(a,z)$ we have $\partial Y(a,z)\Omega\Big|_{z=0} = Ta_{(-n-1)}\Omega = Ta$. By again using the vacuum axiom and furthermore translation covariance, this implies that $\partial Y(a,z)\Omega$ satisfies the initial value problem

$$\begin{aligned} \partial(\partial Y(a,z))\Omega &= [T, \partial Y(a,z)]\Omega = T\partial Y(a,z)\Omega, \\ \partial Y(a,0)\Omega &= Ta. \end{aligned}$$

With lemma 4.6 we conclude $\partial Y(a,z)\Omega = e^{zT}Ta$, which – with corollary 4.5 – finishes the proof. □

Theorem 4.2 (Associativity of the Operator Product Expansion). *Let V be a vertex algebra and $a, b \in V$. For all fields $Y(a,z), Y(b,w)$ we have the associativity relation*

$$Y(a,z)Y(b,w) = Y(Y(a,z-w)b,w) = \sum_{j=0}^{\infty} \iota_{z,w} \frac{Y(a_{(j)}b,w)}{(z-w)^{j+1}} + :Y(a,z)Y(b,w):.$$

With eq. (4.4) this automatically implies that the commutator of those fields is given by

$$[Y(a,z),\, Y(b,w)] = \sum_{j=0}^{\infty} D_w^j \delta(z-w) Y(a_{(j)}b,w).$$

Proof. Let $a, b, c \in V$. We use corollary 4.4 to derive

$$\begin{aligned} Y(a,z)Y(b,w)\Omega &= Y(a,z)e^{wT}b \\ &= e^{wT}Y(a,z-w)e^{-wT}e^{wT}b \\ &= Y(Y(a,z-w)b,w)\Omega, \end{aligned}$$

where this implicitly defines $Y(Y(a,z-w)b,w) := \sum_{n\in\mathbb{Z}} Y(a_{(n)}b,w)(z-w)^{-n-1}$ via the relation

$$\begin{aligned} e^{wT}Y(a,z-w)b &= e^{wT}\sum_{n\in\mathbb{Z}} a_{(n)}b(z-w)^{-n-1} \\ &= \sum_{n\in\mathbb{Z}} Y(a_{(n)}b,w)\Omega(z-w)^{-n-1}. \end{aligned}$$

Let $M, N \in \mathbb{N}$ such that for the field $Y(c,t)$ we have

$$\begin{aligned} (t-z)^M(t-w)^N Y(a,z)Y(b,w)Y(c,t)\Omega &= (t-z)^M(t-w)^N Y(c,t)Y(a,z)Y(b,w)\Omega \\ &= (t-z)^M(t-w)^N Y(c,t)Y(Y(a,z-w)b,w)\Omega \\ &= (t-z)^M(t-w)^N Y(Y(a,z-w)b,w)Y(c,t)\Omega. \end{aligned}$$

Using the locality axiom $Y(c,t)\Omega\Big|_{t=0} = c$ yields

$$z^M w^N Y(a,z)Y(b,w)c = z^M w^N Y(Y(a,z-w)b,w)c.$$

Since c was arbitrary, this proves the first equality in the theorem. For the second one, we use corollary 4.1 to deduce that the commutator between $Y(a,z)$ and $Y(b,w)$ is given by

$$[Y(a,z), Y(b,w)] = \sum_{j=0}^{\infty} c^j D_w^j \delta(z-w),$$

with the coefficients

$$\begin{aligned} c^j(w) &= \mathrm{Res}_{z=w}(z-w)^j Y(Y(a_{(n)}b, z-w), w) \\ &= \mathrm{Res}_{z=w} \sum_{n\in\mathbb{Z}} Y(a_{(n)}b, w)(z-w)^{j-n-1} \\ &= Y(a_{(j)}b, w). \end{aligned}$$

With corollary 4.2 this finally proves the second equality. □

Theorem 4.3. *Let V be a vertex algebra and $v \in V$. The field $L(z) := Y(v,z) = \sum_{n\in\mathbb{Z}} v_{(n)} z^{-n-1} \equiv \sum_{n\in\mathbb{Z}} L^v_{(n)} z^{-n-2}$ is a* VIRASORO *field with central charge c if L^v_{-1} acts as the infinitesimal translation operator T and*

$$\begin{aligned} L_0^v v &\equiv v_{(1)} v = 2v, \\ L_2^v v &\equiv v_{(3)} v = \frac{c}{2}\Omega, \\ L_n^v v &\equiv v_{(n+1)} v = 0 \quad \text{for } n > 2. \end{aligned}$$

The element v is a conformal vector if $L_0^v \equiv v_{(1)}$ is diagonisable (cf. definition 4.10).

Proof. Due to theorem 4.2, the operator product expansion of $L(z)L(w)$ is given by

$$\begin{aligned} L(z)L(w) &\sim \frac{Y(L_{-1}^v v, w)}{z-w} + \frac{Y(L_0^v v, w)}{(z-w)^2} + \frac{Y(L_1^v v, w)}{(z-w)^3} + \frac{Y(L_2^v v, w)}{(z-w)^4} \\ &= \frac{Y(Tv, w)}{z-w} + \frac{2Y(v,w)}{(z-w)^2} + \frac{Y(L_1^v v, w)}{(z-w)^3} + \frac{c}{2}\frac{Y(\Omega, w)}{(z-w)^4}. \end{aligned}$$

Thus, $L(z)$ is a VIRASORO field (cf. definition 4.9) if and only if

i) $Y(Tv, w) = \partial L(w)$,

ii) $Y(v, w) = L(w)$,

iii) $Y(L_1^v v, w) = 0$,

iv) $Y(\Omega, w) = \mathbb{1}_V$.

While the first condition follows immediately with corollary 4.6, the second and fourth point is the definition of $L(z)$ resp. of the vertex algebra V. It remains to show that the term containing the field $\alpha(w) := Y(L_1^v v, w)$ vanishes. Since the field $L(z) = Y(v,z)$ is local with respect to itself, the products $L(z)L(w)$ and $L(w)L(z)$ must be expansions for $|z| > |w|$ resp. $|z| < |w|$ of the same element in $\mathrm{End}(V)((z,w))$. This is equivalent to saying, their operator product expansions are equal. Interchanging z and w in the operator product expansion above and using TAYLOR's theorem yields that

$$\begin{aligned} L(w)L(z) &\sim \frac{\partial L(z)}{w-z} + \frac{2L(z)}{(w-z)^2} + \frac{\alpha(z)}{(w-z)^3} + \frac{c}{2}\frac{1}{(w-z)^4} \\ &= -\frac{\partial L(w)}{z-w} + \frac{2L(w) + 2(z-w)D_w L(w)}{(z-w)^2} - \frac{\alpha(w) + (z-w)D_w\alpha(w) + (z-w)^2 D_w^2 \alpha(w)}{(z-w)^3} + \frac{c}{2}\frac{1}{(z-w)^4}, \end{aligned}$$

where we used that the operator expansion contains only terms of negative order in $(z-w)$. Comparing the coefficients of $(z-w)^{-3}$ in both operator product expansions finally yields $\alpha(w) = 0$. □

At the end of section 4.3, we mentioned that the HEISENBERG vertex algebra S contains a one-parameter family of conformal vectors. We claim that these vectors are given by

$$v_\lambda = \frac{1}{2} T_1^2 + \lambda T_2, \quad \lambda \in \mathbb{C}.$$

We define the fields $Y(v_\lambda, z) := \sum_{n\in\mathbb{Z}} L_n^\lambda z^{-n-2}$ or equivalently $L_n^\lambda := (v_\lambda)_{(n)}$. According to definition 4.10 we have to verify that L_{-1}^λ acts as an infinitesimal translation operator T, that L_0^λ is diagonisable, and that $Y(v_\lambda, z)$ is a VIRASORO field. First, we develop an explicit formula for the coefficients L_n^λ by expanding the fields $Y(T_1^2, z)$ and $Y(T_2, z)$. With the notation of eq. (4.13) we have with $\rho(a_0) = 0$.

$$\begin{aligned} Y(T_1^2, z) &= \ :A(z)A(z): \\ &= \sum_{k\neq 0} \sum_{m+n=k} a_m a_n z^{-k-2} + 2 \sum_{m\in\mathbb{N}} a_{-m} a_m z^{-2}, \\ Y(T_2, z) &= \partial A(z) \\ &= \sum_{k\in\mathbb{Z}} (-k-1) a_k z^{-k-2}. \end{aligned}$$

Combining these equations yields

$$\begin{aligned} Y(v_\lambda, z) &= \frac{1}{2} \sum_{k\neq 0} \left(\sum_{m+n=k} a_m a_n - 2\lambda(k+1) a_k \right) z^{-k-2} + \sum_{m\in\mathbb{N}} a_{-m} a_m z^{-2} \\ \Longrightarrow L_0^\lambda &= \sum_{m\in\mathbb{N}} a_{-m} a_m \quad \text{and} \quad L_{-1}^\lambda = \sum_{m\in\mathbb{N}} a_{-m-1} a_m. \end{aligned}$$

With the same decomposition as before, we can split the space S into eigenspaces of L_0^λ with eigenvalue $\mathbf{deg}(P)$ for $P \in S$. We have shown in eq. (4.12) that L_0^λ defined in the way above acts on S as an infinitesimal translation operator T. We now want to apply theorem 4.3 with $V = S$ to show that the fields $Y(v_\lambda, z)$ are indeed VIRASORO fields. It remains to show that $L_2^\lambda v_\lambda = \frac{c}{2}\Omega$ for some $c \in \mathbb{C}$, that $L_0^\lambda v_\lambda = 2v$, and that $L_n^\lambda v_\lambda = 0$. We calculate

$$\begin{aligned} L_2^\lambda v_\lambda &= \frac{1}{2} \left(\sum_{m\in\mathbb{Z}} a_m a_{2-m} - 6\lambda a_2 \right) v_\lambda \\ &= \frac{1}{2} \left(\sum_{m\geq 3} a_m (2-m) T_{m-2} v_\lambda + \sum_{m\leq 1} a_m \partial_{2-m} v_\lambda \right) - 3\lambda^2 \Omega \\ &= \frac{1}{2} \partial_1 T_1 - 3\lambda^2 \Omega \\ &= \frac{1}{2} \left(1 - 6\lambda^2\right) \Omega \end{aligned}$$

and conclude that we should set $c = 1 - 6\lambda^2$. For $n > 2$ we have

$$\begin{aligned} L_n^\lambda v_\lambda &= \frac{1}{2} \sum_{m\in\mathbb{Z}} a_m a_{n-m} v_\lambda \\ &= \frac{1}{2} (a_{n-1} a_1 v_\lambda + a_{n-2} a_2 v_\lambda) \\ &\propto \partial_{n-1} T_1 + \partial_{n-2} \lambda \Omega = 0. \end{aligned}$$

Finally,

$$\begin{aligned} L_0^\lambda v_\lambda &= \sum_{m\in\mathbb{N}} a_{-m} a_m v_\lambda \\ &= a_{-1} T_1 + a_{-2} \lambda \Omega \\ &= T_1^2 + 2\lambda T_2 = 2 v_\lambda. \end{aligned}$$

Now, all the conditions of theorem 4.3 are fulfilled and we conclude that v_λ is a conformal vector with central charge $c = 1 - 6\lambda^2$.

Chapter 5

Superconformal Vertex Algebras and their Role in Superstring Theory

The following two sections should be seen as a (very) brief introduction to the idea of supersymmetry in string theory. Their structure and some results are taken from the fourth chapter in [2].

In conformal gauge, the bosonic theory in D spacetime dimensions is described by the action on the world-sheet Σ

$$S[X] = -\frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\ \partial_\alpha X_\mu \partial^\alpha X^\mu,$$

where $\alpha, \mu \in \{0, \dots, D-1\}$. We can generalise this theory by introducing a D-plet of free two-dimensional fermionic fields Ψ^μ_A to the action

$$S[X,\psi] = -\frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\ \left(\partial_\alpha X_\mu \partial^\alpha X^\mu - \imath\overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right), \tag{5.1}$$

where ρ^α denotes for $\alpha \in \{0,1\}$ a set of generators of the two-dimensional Clifford algebra. We will use the convention

$$\rho^0 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} \quad \text{and} \quad \rho^1 = \begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix},$$

yielding the reality of the Dirac operator $\imath\rho^\alpha\partial_\alpha$. Omitting the spacetime index μ, we can assume that Ψ describes a Majorana particle given by

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}.$$

5.1 World-Sheet Supersymmetry

Supersymmetry, a symmetry between the fields X^μ and ψ^μ, can be made manifest by extending the two-dimensional world-sheet Σ by adding two Grassmann coordinates to the system. The two original coordinates will be denoted by σ^α, whereas the two odd coordinates will be referred to by θ^A. A general function on this so called superspace $\hat{\Sigma}$ can be written as

$$Y^\mu(\sigma,\theta) = X^\mu(\sigma) + \overline{\theta}\psi^\mu(\sigma) + \frac{1}{2}\overline{\theta}\theta B^\mu(\sigma),$$

for some field $B(\sigma)$. On $\hat{\Sigma}$, Supersymmetry is generated by the operator

$$Q_A := \frac{\partial}{\partial\overline{\theta}} + \imath\left(\rho^\alpha\theta\right)_A \partial_\alpha.$$

To investigate the transformation of the coordinates under the generator Q_A, we contract it with an infinitesimal anti-commuting vector ϵ_A, yielding

$$\begin{aligned}
\delta\theta^A &= \left[\bar{\epsilon}_B Q^B, \theta^A\right] = \left[\overline{O}^B \epsilon_B, \theta^A\right] = \epsilon^A, \\
\delta\sigma^\alpha &= \left[\bar{\epsilon}_B Q^B, \sigma^\alpha\right] = \bar{\epsilon}_B \imath (\rho^\beta \theta)^B \partial_\beta \sigma^\alpha = \imath \epsilon \rho^\alpha \theta.
\end{aligned}$$

With the relation[1]

$$\theta_A \overline{\theta}_B = -\frac{1}{2}\delta_{AB}\overline{\theta}_C \theta_C$$

we derive the transformation of the general field $Y^\mu(\sigma,\theta)$ under $\bar{\epsilon}Q$[2]

$$\begin{aligned}
\delta Y^\mu(\sigma,\theta) &= \left[\bar{\epsilon}Q,\, Y^\mu\right] \\
&= \bar{\epsilon}QX^\mu(\sigma) + \bar{\epsilon}Q\overline{\theta}\psi^\mu(\sigma) + \bar{\epsilon}Q\frac{1}{2}\overline{\theta}\theta B^\mu(\sigma) \\
&= \imath\bar{\epsilon}\rho^\alpha\theta\partial_\alpha X^\mu(\sigma) + \bar{\epsilon}\psi^\mu(\sigma) + \imath\left(\bar{\epsilon}\rho^\alpha\theta\right)\partial_\alpha\overline{\theta}\psi(\sigma)^\mu + \bar{\epsilon}\theta B^\mu(\sigma) \\
&= \bar{\epsilon}\psi^\mu(\sigma) + \overline{\theta}\left(-\imath\rho^\alpha\epsilon\partial_\alpha X(\sigma)^\mu + \epsilon B^\mu(\sigma)\right) + \frac{1}{2}\overline{\theta}\theta\left(-\imath\bar{\epsilon}\rho^\alpha\partial_\alpha\psi^\mu(\sigma)\right).
\end{aligned}$$

This is equivalent to saying that the fields X^μ, ψ^μ, and B^μ transform as

$$\begin{aligned}
\delta X^\mu &= \bar{\epsilon}\psi^\mu, \\
\delta\psi^\mu &= -\imath\rho^\alpha\epsilon\partial_\alpha X^\mu + B^\mu\epsilon, \\
\delta B^\mu &= -\imath\bar{\epsilon}\rho^\alpha\partial_\alpha\psi^\mu.
\end{aligned} \tag{5.2}$$

In the following, we want to show that the action in eq. (5.1) can also be written as

$$\frac{\imath}{4\pi}\int_{\hat{\Sigma}} \mathrm{dvol}\; DY^\mu \overline{D} Y_\mu, \tag{5.3}$$

where D is the *superspace covariant derivative* given by

$$D = \frac{\partial}{\partial\overline{\theta}} - \imath\rho^\alpha\theta\partial_\alpha.$$

[1]With $\overline{\theta} = \theta^T\rho^0$ we have for the components $\overline{\theta}_1 = \imath\theta_2$ and $\overline{\theta}_2 = -\imath\theta_1$. Thus,

$$\begin{aligned}
\theta_1\overline{\theta}_2 &= -\imath\theta_1^2 = 0 \\
\theta_2\overline{\theta}_1 &= \imath\theta_2^2 = 0
\end{aligned}$$

and furthermore $\theta_1\overline{\theta}_1 = \theta_2\overline{\theta}_2$. The result follows with $\overline{\theta}\theta = -2\theta_1\overline{\theta}_1$.

[2]The last equality holds, since (the upper (lower) sign describes the case $\alpha = 0$ ($\alpha = 1$))

$$\begin{aligned}
\left(\bar{\epsilon}\rho^\alpha\theta\right)\overline{\theta}\psi &= \epsilon^T\rho^0\rho^\alpha\theta\theta^T\rho^0\psi \\
&= (\epsilon_1\theta_1 \pm \epsilon_2\theta_2)\left(-\imath\theta_1\psi_2 + \imath\theta_2\psi_1\right) \\
&= \imath\epsilon_1\theta_2\psi_1 \pm \imath\epsilon_2\theta_1\theta_2\psi \\
&= \imath\theta_1\theta_2\epsilon^T\rho^0\rho^\alpha\psi \\
&= -\frac{1}{2}\overline{\theta}\theta\bar{\epsilon}\rho^\alpha\psi.
\end{aligned}$$

Applying this derivative to some superfield $Y^\mu(\sigma,\theta)$ yields[3]

$$\begin{aligned} DY^\mu &= Y^\mu + \theta B^\mu - \imath\rho^\alpha\theta\partial_\alpha X^\mu - \imath\partial_\alpha\rho^\alpha\theta\left(\overline{\theta}\psi^\mu\right) \\ &= Y^\mu + \theta B^\mu - \imath\rho^\alpha\theta\partial_\alpha X^\mu + \frac{\imath}{2}\overline{\theta}\theta\rho^\alpha\partial_\alpha\psi^\mu \\ \overline{D}Y^\mu &= \overline{\psi}^\mu + B^\mu\overline{\theta} + \imath\partial_\alpha X^\mu\overline{\theta}\rho^\alpha - \frac{\imath}{2}\overline{\theta}\theta\partial_\alpha\overline{\psi}^\mu\rho^\alpha. \end{aligned}$$

The integration over the superspace includes two integrals over the two GRASSMANN variables. We use the definition of the BEREZIN integral, namely

$$\int d^2\theta \; \left(a + \theta^1 b_1 + \theta^2 b_1 + \theta^1\theta^2 c\right) = c.$$

Using partial integration, we furthermore find

$$\int d^2\theta \frac{\partial V}{\partial\theta^A} = 0$$

for any function $V(\theta)$. Considering eq. (5.3), the terms of the integrand that are linear in $\overline{\theta}\theta$ and thus remain after the integration are

$$B^\mu B_\mu\overline{\theta}\theta + \partial_\alpha X^\mu\partial_\beta X_\mu\overline{\theta}\underbrace{\rho^\alpha\rho^\beta}_{\substack{=-\frac{1}{2}\{\rho^\alpha,\rho^\beta\} \\ =-\eta^{\alpha\beta}}}\theta + \frac{\imath}{2}\overline{\theta}\theta\underbrace{\left(\overline{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu - \partial_\alpha\overline{\psi}^\mu\rho^\alpha\psi_\mu\right)}_{2\overline{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu},$$

where we used partial integration for the last term. With $\overline{\theta}\theta = -2\imath\theta_1\theta_2$ we find

$$S[X,\psi] = (-2\imath)\frac{\imath}{4\pi}\int_\Sigma d^2\sigma \; \left(-\partial_\alpha X_\mu\partial^\alpha X^\mu + \imath\overline{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu + B^\mu B_\mu\right).$$

The variations with respect to B^μ yield the equations of motion $B^\mu = 0$ and we recover the integral in eq. (5.3). Let us now assume that ϵ, the infinitesimal parameter of the transformation in eq. (5.2), is not constant. When applying the transformation to the action and using the equations of motion $B^\mu = 0$, $\partial^2 X = 0$, and $\rho^\alpha\partial_\alpha\psi = 0$, we receive

$$\begin{aligned} 0 \stackrel{!}{=} \delta S[X,\psi] &= -\frac{1}{2\pi}\int_\Sigma d^2\sigma \; \left(2\partial_\alpha X_\mu\partial^\alpha(\delta X^\mu) - \imath\overline{\psi}^\mu\rho^\alpha\partial_\alpha(\delta\psi_\mu) - \imath(\delta\overline{\psi}^\mu)\rho^\alpha\partial_\alpha\psi_\mu\right) \\ &= -\frac{1}{2\pi}\int_\Sigma d^2\sigma \; \left(-2\partial^2 X_\mu(\delta X^\mu) - \imath\overline{\psi}^\mu\rho^\alpha\partial_\alpha(-\imath\rho^\beta\epsilon\partial_\beta X_\mu)\right) \\ &= \frac{1}{2\pi}\int_\Sigma d^2\sigma \; \left(\overline{\psi}^\mu\rho^\alpha\rho^\beta\partial_\alpha\epsilon\partial_\beta X_\mu + \overline{\psi}^\mu\rho^\alpha\rho^\beta\epsilon\partial_\alpha\partial_\beta X_\mu\right) \\ &= \frac{1}{2\pi}\int_\Sigma d^2\sigma\left(\partial_\alpha\overline{\epsilon}\right)J^\alpha, \end{aligned}$$

where we defined the *supercurrent*

$$J^\alpha := \overline{\psi}^\mu\rho^\beta\rho^\alpha\psi^\mu\partial_\beta X_\mu, \tag{5.4}$$

[3]The second equality holds due to

$$\begin{aligned} \rho^\alpha\theta\left(\theta^T\rho^0\psi\right) &= \rho^\alpha\theta\left(-\imath\theta_1\psi_2 + \imath\theta_2\psi_1\right) = \rho^\alpha\theta\begin{pmatrix}\imath\theta_1\theta_2\psi_1 \\ -\imath\theta_2\theta_1\psi_2\end{pmatrix} \\ &= \rho^\alpha\begin{pmatrix}\theta_1\overline{\theta}_1\psi_1 \\ \theta_2\overline{\theta}_2\psi_2\end{pmatrix} = -\frac{1}{2}\overline{\theta}\theta\rho^\alpha\psi. \end{aligned}$$

which is conserved on the mass-shell. Furthermore, one finds that the energy-momentum tensor of the system is given by [4]

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu + \frac{\imath}{4}\overline{\psi}^\mu \rho_a \partial_b \psi_\mu + \frac{\imath}{4}\overline{\psi}^\mu \rho_b \partial_a \psi_\mu - (\text{trace}).$$

In light-cone coordinates (see below) this expression reduces to

$$\begin{aligned} T_{++} &= \partial_+ X^\mu \partial_+ X_\mu + \frac{\imath}{2}\psi^\mu_+ \partial_+ \psi_{+\mu}, \\ T_{--} &= \partial_- X^\mu \partial_- X_\mu + \frac{\imath}{2}\psi^\mu_- \partial_- \psi_{-\mu}. \end{aligned} \tag{5.5}$$

5.2 NEVEU-SCHWARZ and RAMOND Boundary Conditions

For the bosonic fields X^μ, we obtain the usual boundary conditions, i.e. DIRICHLET and NEUMANN conditions for open strings, and periodicity for closed strings. As in section 1.3, we introduce light-cone coordinates as $x_\pm = x^0 \pm x^1$, resp. $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$. The equations of motion for the fermionic field ψ can be written as

$$0 = \rho^\alpha \partial_\alpha \psi = \imath \begin{pmatrix} 0 & -\partial_0 + \partial_1 \\ \partial_0 + \partial_1 & 0 \end{pmatrix} \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}.$$

Thus, the system is described by the constraint equations

$$\partial_\pm \psi^\mu_\mp = \partial_\pm \left(\partial_\mp X^\mu\right) = 0.$$

Similar to the bosonic case, the boundary conditions for the field ψ in the case of an open string originate in the term arising in the derivation of the equations of motion from the action due to partial integration. We assume that the variation vanishes at the endpoints of integration in the direction of σ^0. Then, the term arising from partial integration is given by

$$\begin{aligned} \int_\Sigma \mathrm{d}^2\sigma\ \partial_\alpha \left(\overline{\psi}^\mu \rho^\alpha \delta\psi_\mu\right) &= \int_\Sigma \mathrm{d}^2\sigma \left(\partial_0 \left(\overline{\psi}^\mu \rho^0 \delta\psi_\mu\right) + \partial_1 \left(\overline{\psi}^\mu \rho^1 \psi_\mu\right)\right) \\ &= \int_{\sigma^0_0}^{\sigma^0_1} \mathrm{d}\sigma^0 \left(\overline{\psi}\rho^1 \delta\psi\right)\Big|_{\sigma^1=0}^{\sigma^1=\pi}. \end{aligned}$$

Thus, we should impose the boundary condition $0 \overset{!}{=} \overline{\psi}\rho^1\delta\psi = \psi_-\delta\psi_- - \psi_+\delta\psi_+$, satisfied by $\psi_+ = \pm\psi_-$ at the endpoints. Without loss of generality, we can assume that

$$\psi^\mu_+(\sigma^0, 0) = \psi^\mu_-(\sigma^0, 0),$$

since the opposite-sign case is equivalent. Now, either the functions are also equal at $\sigma^1 = \pi$ (RAMOND), or they differ by a sign (NEVEU-SCHWARZ). In both cases, we can expand the functions using FOURIER modes

$$\begin{aligned} \psi^\mu_\pm(\sigma) &= \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} b^\mu_n e^{-\imath n(\sigma^0 \pm \sigma^1)} && \text{(RAMOND)} \\ \psi^\mu_\pm(\sigma) &= \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+1/2} b^\mu_r e^{-\imath r(\sigma^0 \pm \sigma^1)} && \text{(NEVEU-SCHWARZ).} \end{aligned}$$

[4] To derive the energy-momentum tensor, one has to calculate the variation of the action S with respect to a general auxiliary metric h^{cd} and divide by $\sqrt{-h}$. Without the use of the conformal gauge, the action is of the form

$$S[X,\psi] = \frac{1}{2\pi}\int_\Sigma \sqrt{-h}h^{ab}\left(\partial_a X^\mu \partial_b X_\mu + \overline{\psi}\rho_a \partial_b \psi\right).$$

5.3 Super-Virasoro Vertex Algebra

In analogy to the theory of bosons, we again define the operators L_m as the Fourier modes of the energy momentum tensor T_{ab} (cf. eq. (5.5))

$$L_m = \frac{1}{\pi}\int_0^\pi \mathrm{d}\sigma^1 \left(e^{\imath m\sigma^1} T_{++} + e^{-\imath m\sigma^1} T_{--}\right) = \frac{1}{\pi}\int_{-\pi}^\pi \mathrm{d}\sigma^1 e^{\imath m\sigma^1} T_{++},$$

which obviously differs from the bosonic theory, since the energy momentum tensor was extended by the fermionic term. These operators form the bosonic generators of the super-Virasoro algebra. The fermionic operators are defined as the Fourier modes of the supercurrent J (cf. eq. (5.4))

$$G_q = \frac{\sqrt{2}}{\pi}\int_0^\pi \mathrm{d}\sigma^1 \left(e^{\imath q\sigma^1} J_+ + e^{-\imath q\sigma^1} J_-\right) = \frac{\sqrt{2}}{\pi}\int_{-\pi}^\pi \mathrm{d}\sigma^1 e^{\imath q\sigma^1} J_+, \tag{5.6}$$

where $q \in \mathbb{Z}$ for Ramond boundary conditions and $q \in \mathbb{Z}+1/2$ for Neveu-Schwarz boundary conditions. We quantise the system by imposing the equal-time (anti-)commutation relations of the fields as

$$\begin{aligned}\left[\dot{X}^\mu(\sigma^0,x),\, X^\nu(\sigma^0,y)\right] &= -\imath\pi\delta(x-y)\eta^{\mu\nu},\\ \left\{\psi^\mu_A(\sigma^0,x),\, \psi^\nu_B(\sigma^0,y)\right\} &= \delta_{AB}\pi\delta(x-y)\eta^{\mu\nu},\end{aligned}$$

yielding the (anti-)commutation relations of the operators

$$\begin{aligned}\left[a^\mu_m,\, a^\nu_n\right] &= m\eta^{\mu\nu}\delta_{m+n},\\ \left\{b^\mu_m,\, b^\nu_n\right\} &= \eta^{\mu\nu}\delta_{m+n},\\ \left\{b^\mu_r,\, b^\nu_s\right\} &= \eta^{\mu\nu}\delta_{r+s},\end{aligned} \tag{5.7}$$

for $m,n \in \mathbb{Z}$ and $r,s \in \mathbb{Z}+1/2$. With eq. (5.6), we find that the super-Virasoro algebra has a representation

$$\begin{aligned}L_m &= \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{-n}\cdot\alpha_{m+n}: + \frac{1}{2}\sum_q \left(q+\frac{m}{2}\right) :b_{-q}\cdot b_{m+q}:,\\ G_q &= \sum_{n\in\mathbb{Z}} a_{-n}\cdot b_{q+n},\end{aligned}$$

where again $q \in \mathbb{Z}$ in the Neveu-Schwarz sector and $q \in \mathbb{Z}+1/2$ for Ramond boundary conditions. Finally, eq. (5.7) yields the super-Virasoro algebra [5]

$$\begin{aligned}\left[L_m, L_n\right] &= (m-n)L_{m+n} + \frac{D}{8}(m^3-m)\delta_{m+n},\\ \left[L_m, G_q\right] &= \left(\frac{m}{2}-q\right)G_{m+q},\\ \left\{G_p, G_q\right\} &= 2L_{p+q} + \frac{D}{2}\left(q^2-\frac{1}{4}\right)\delta_{p+q}.\end{aligned}$$

Neveu-Schwarz To construct a vertex algebra for $q \in \mathbb{Z}+1/2$, we define a representation space V with the basis

$$\left\{ v_{n_1\ldots n_k} \;\middle|\; 1 < n_k \leq \ldots \leq n_1,\; 2n_i, k \in \mathbb{N} \right\} \cup \{\Omega\}$$

[5] To be precise, the commutation relations in eq. (5.7) yield for the Ramond operators anomalous terms of the form $\frac{D}{8}m^3$ and $\frac{D}{2}q^2$. To generalise the theory and unite both sectors one has to shift the operator $L_0 \to L_0 - \frac{1}{2}\frac{D}{8}$ in this sector.

While the representation of the even part of the algebra, consisting of the L_ns, is still given by eq. (4.15), we define the action of the odd sector as

$$G_q\Omega := \begin{cases} 0 & \text{for } q > -\frac{3}{2} \\ v_{-q} & \text{for } q \leq -\frac{3}{2} \end{cases}$$
$$G_{-q} v_{n_1\ldots n_k} = v_{qn_1\ldots n_k} \quad \text{for } q > n_1.$$

We introduce the operator $T := L_{-1}$ and the fields

$$L(z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2},$$
$$G(z) := \sum_{r\in\mathbb{Z}+1/2} G_r z^{-r-3/2}.$$

We note that this definition is in accordance with eq. (4.14) and definition 4.6, since, with the grading of V via eigenvalues of L_0, $G_{(n)} = G_{n-1/2}$ is homogeneous of degree $-n+1/2 = 3/2-n-1$. Furthermore, we define

$$a := L(z)\Omega\Big|_{z=0} = L_{-2}\Omega = v_2,$$
$$b := G(z)\Omega\Big|_{z=0} = G_{-3/2}\Omega = v_{3/2}.$$

We want to show that these fields are local to each other and to themselves, i.e. that the corresponding commutator is a linear combination of derivatives of the delta distribution. We already proved the locality of $L(z)$ with respect to itself in eq. (4.16). For $G(z)$ we calculate with $s = r+n$

$$\begin{aligned}[]
[L(z), G(w)] &= \sum_{n,r} [L_n, G_r]\, z^{-n-2} w^{-r-3/2} \\
&= \sum_{n,r} \left(\frac{n}{2} - r\right) G_{n+r} z^{-n-2} w^{-r-3/2} \\
&= \sum_{n,s} \left(\frac{3}{2}n - s\right) G_s z^{-n-2} w^{-s-3/2} w^n \\
&= \sum_n \frac{3}{2} n z^{-n-2} w^n \sum_s G_s w^{-s-3/2} - \sum_n z^{-n-1} w^n \sum_s s G_s w^{-s-5/2} \\
&= \frac{3}{2} G(w) \partial_w \delta(z-w) + \partial_w G(w) \delta(z-w)
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
\{G(z), G(w)\} &= \sum_{r,s} \{G_r, G_s\}\, z^{-r-3/2} w^{-s-3/2} \\
&= \sum_{rs} 2L_{r+s} z^{-r-3/2} w^{-s-3/2} + \sum_r \frac{D}{2} \underbrace{\left(r^2 - \frac{1}{2}\right)}_{=(r+1/2)(r-1/2)} z^{-r-3/2} w^{r-3/2} \\
&= 2L(z)\delta(z-w) \frac{D}{2} \partial_w^2 \delta(z-w).
\end{aligned}$$

This implies with corollary 4.1 locality and yields with $\partial^2 = 2D^2$ the operator product expansion

$$L(z)G(w) \sim \frac{D}{(z-w)^3} + \frac{2L(w)}{z-w},$$
$$G(z)G(w) \sim \frac{3}{2} \frac{G(w)}{(z-w)^2} + \frac{\partial_w G(w)}{z-w}.$$

Finally, the relations

$$[T, G(z)] = \sum_r [L_{-1}, G_r]\, z^{-r-3/2} = \sum_r \left(-\frac{1}{2} - r\right) G_{r-1} z^{-r-3/2} = \partial G(z),$$
$$[T, L(z)] = \sum_n [L_{-1}, L_n]\, z^{-n-2} = \sum_n (-1-n) L_{n-1} z^{-n-2} = \partial L(z)$$

and $T\Omega = L_{-1}\Omega = 0$ allow us to apply theorem 4.1, which yields a vertex algebra structure on V.

RAMOND Two problems arise in the development of a vertex algebra structure in the RAMOND sector. The first one is a degeneracy of the vacuum Ω, due to the commutation of the operators b_0^μ with the mass operator M given by $\alpha' M^2 = L_0 + \text{const.}$. Thus, all states of the form

Vacuum element	Multiplicity
Ω	1
$b_0^{\mu_1}\Omega$	D
$b_0^{\mu_1} b_0^{\mu_2}\Omega$	$\frac{D(D-1)}{2}$
$\vdots$	$\vdots$
$b_0^{\mu_1} \dots b_0^{\mu_D}\Omega$	1

with $\mu_i \neq \mu_j$ for $i, j \in \{1, \dots, D\}$ are part of the same eigenspace of M^2. This yields a total degeneracy of

$$\sum_{n=0}^{D} \frac{D!}{(D-n)!}\frac{1}{n!} = \sum_{n=0}^{D} \binom{D}{n} = 2^D.$$

The second problem comes from the vacuum axiom of a vertex algebra, namely

$$Y(a, z)\Omega|_{z=0} = a.$$

Assuming that the field $L(z)$ is given by the same expansion as in the NEVEU-SCHWARZ sector, we can show with an analogous calculation as we performed in eq. (5.8) that $G(z)$ has to be of the form

$$G(z) = \sum_{n\in\mathbb{Z}} G_n z^{-n-3/2}$$

for it to be local with respect to $L(z)$. When choosing a different exponent of z, the commutator between $L(z)$ and $G(z)$ can no longer be written as a linear combination of derivatives of the delta distribution. Now, the vacuum axiom requires $G_n\Omega = 0$ for all $n \geq -1$. With the commutation relations of the operators G_n we receive

$$0 = \{G_1, G_{-1}\}\,\Omega = 2L_0\Omega + \frac{3D}{8}\Omega.$$

On the other hand, a representation with $L_0\Omega \neq 0$ is in disagreement with the vacuum axiom for $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$.

We conclude that while the construction of the NEVEU-SCHWARZ vertex algebra is possible without any difficulties, the RAMOND vertex algebra at least requires several deviations from the vertex algebra axioms we imposed in this work.

Bibliography

[1] V. Kac,
Vertex Algebras for Beginners, ed.2,
1998.

[2] M.B. Green, J.H. Schwarz & E. Witten,
Superstring Theory, Volume 1,
1988.