# Light Cone Gauge for Bosonic Strings

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## Contents



# 1 Introduction

In following, we will examine the behaviour of relativistic bosonic strings embedded in $D = d + 1$ dimensional spacetime. After introducing a suitable action, we will derive the Lorentz generators for the relativistic string. Then, we will introduce the *light cone gauge*, in which we will quantise the relativistic string and finally conclude that for bosonic string theory to be Lorentz invariant, the number of spacetime dimensions is restricted to $D = 26$. Throughout the report we will use Einstein's summation convention and a spacetime metric with signature $(\rho_-, \rho_+) = (1, d)$.

This report is based on '*A First Course in String Theory*' by Barton Zwiebach [1]. Additional information was acquired from the lecture notes '*Introduction to String Theory*' of Christoph A. Keller [2].

## 2 Relativistic String

### 2.1 Area Functional for Spacetime Surfaces

Suppose we have an infinitesimal area, formed by the two vectors $\mathbf{dv}_1$ and $\mathbf{dv}_2$, which themselves are of infinitesimal length. By defining $\theta$ as the angle between the vectors we derive the area of the parallelogram as

$$\begin{aligned} \mathrm{d}A &= |\mathbf{dv}_1||\mathbf{dv}_2|\sin(\theta) \\ &= |\mathbf{dv}_1||\mathbf{dv}_2|\sqrt{1-\cos^2(\theta)} \\ &= |\mathbf{dv}_1||\mathbf{dv}_2|\sqrt{1-\left(\frac{\mathbf{dv}_1\cdot\mathbf{dv}_2}{|\mathbf{dv}_1||\mathbf{dv}_2|}\right)^2} \\ &= \sqrt{|\mathbf{dv}_1|^2|\mathbf{dv}_2|^2-(\mathbf{dv}_1\cdot\mathbf{dv}_2)^2}. \end{aligned}$$

Since we are interested in the behaviour of a string – namely a compact, path-connected, one-dimensional subset of space – we are dealing with a two dimensional surface in spacetime, bounded by the endpoints of the string. For each dimension of the surrounding spacetime in which the two dimensional surface is embedded, we therefore want to introduce a mapping function $X^\mu:\ \mathbb{R}\times[0,\sigma_1]\to\mathbb{R}$. We will call the first argument $\tau$ and the second one $\sigma$, corresponding to time respectively space.

Using the infinitesimal change in the direction of $\tau$ resp. $\sigma$ as the vectors $\mathbf{dv}_1$ resp. $\mathbf{dv}_2$

$$\mathrm{d}\mathbf{v}_1^\mu = \frac{\partial X^\mu}{\partial\tau}\mathrm{d}\tau, \qquad \mathrm{d}\mathbf{v}_2^\mu = \frac{\partial X^\mu}{\partial\sigma}\mathrm{d}\sigma,$$

one could assume that the infinitesimal area of the world-sheet is given by

$$\mathrm{d}A \overset{?}{=} \sqrt{\left(\frac{\partial X}{\partial\tau}\right)^2\left(\frac{\partial X}{\partial\sigma}\right)^2-\left(\frac{\partial X^\mu}{\partial\tau}\cdot\frac{\partial X_\mu}{\partial\sigma}\right)^2}.$$

We will now show, that the sign beneath the square root of the expression above is negative, whenever there are both a timelike and a spacelike direction on the world-sheet.

*Proof.* Consider the set of all tangent directions at a given point p on the world-sheet

$$\mathbf{v}^\mu(\lambda) = \frac{\partial X^\mu}{\partial\tau}+\lambda\frac{\partial X^\mu}{\partial\sigma}, \qquad \lambda\in[-\infty,\infty].$$

To ensure that the set contains both spacelike and timelike vecors, we define the polynomial $p(\lambda)$

$$p(\lambda) := \mathbf{v}^\mu(\lambda)\,\mathbf{v}_\mu(\lambda) = \lambda^2\left(\frac{\partial X}{\partial\sigma}\right)^2+2\lambda\left(\frac{\partial X^\mu}{\partial\tau}\cdot\frac{\partial X_\mu}{\partial\sigma}\right)+\left(\frac{\partial X}{\partial\tau}\right)^2$$

and require $p$ to take both positive and negative values. For this to hold, the discriminant in the quadratic equation $p(\lambda)=0$ has to be positive, ergo

$$\left(\frac{\partial X^\mu}{\partial\tau}\cdot\frac{\partial X_\mu}{\partial\tau}\right)^2-\left(\frac{\partial X}{\partial\tau}\right)^2\left(\frac{\partial X}{\partial\sigma}\right)^2>0.$$

□

Thus, the total area is given by

$$A=\int\mathrm{d}\tau\int\mathrm{d}\sigma\sqrt{-g},$$

where $g=\det(g_{\mu\nu})$ is the determinant of the induced metric

$$g_{\mu\nu}=\frac{\partial X}{\xi^\mu}\cdot\frac{\partial X}{\xi^\nu}.$$

## 2.2 The Nambo-Goto String Action

Motivated by the derivation of the infinitesimal surface area in spacetime in the last section, we define the string action as

$$S = -T_0\, A = -T_0 \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma\, \sqrt{\left(\dot{X}\cdot X'\right)^2 - \dot{X}^2 X'^2} \tag{1a}$$

$$= -T_0 \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma \sqrt{-g} \tag{1b}$$

where $T_0$ is the string tension and $\dot{X}^\mu := \frac{\partial X^\mu}{\partial \tau}$ and $X^{\mu\,\prime} := \frac{\partial X^\mu}{\partial \sigma}$. This is called the Nambo-Goto-action, named after Yoichiro Nambo and Tetsuo Goto. For completeness, we note that this action is equivalent to the Polyakov action, given by

$$S_P = -\frac{T_0}{2} \int \mathrm{d}\tau \int \mathrm{d}\sigma \sqrt{-h}\, h^{ab}\, \partial_a X^\mu\, \partial_b X_\mu,$$

where we introduced the world-sheet metric $h_{ab}$, its determinant $h$ and its inverse $h^{ab}$. Note that the summation over the indices $a$ and $b$ is taken over $\{\tau, \sigma\}$, whereas the summation in $\mu$ and $\nu$ is over the $D$ dimensions of the surrounding space.

*Proof.* We first derive the equations of motion for the world-sheet metric $h_{ab}$ induced by the Polyakov action and use the result to verify the equivalence to the Nambo-Goto-action. Since the action is independent of the derivatives of $h_{ab}$ we receive from the Euler-Lagrange equation

$$\begin{aligned} 0 \overset{!}{=} -\frac{2}{T_0}\frac{\delta \mathscr{L}_P}{\delta h^{cd}} &= \frac{\partial}{\partial h^{cd}}\left(\sqrt{-h}\, h^{ab}\, \partial_a X^\mu\, \partial_b X_\mu\right) \\ &= \frac{\partial \sqrt{-h}}{\partial h^{cd}} h^{ab} \partial_a X^\mu \partial_b X_\mu + \sqrt{-h}\, \partial_c X^\mu \partial_d X_\mu \end{aligned}$$

Note that the derivative is with respect to the inverse of $h_{ab}$. Thus, to be able to use the special case of Jacobi's formula $\frac{\partial \det(A)}{\partial A_{ij}} = \mathrm{adj}(A)_{ij}$ we first define the determinant of $h^{ab}$ as $\tilde{h} := \frac{1}{h}$. We calculate

$$\begin{aligned} \frac{\partial \sqrt{-h}}{\partial h^{cd}} h^{ab} \partial_a X^\mu \partial_b X_\mu + \sqrt{-h}\, \partial_c X^\mu \partial_d X_\mu &= \frac{\partial}{\partial h^{cd}}\left(\frac{1}{\sqrt{-\tilde{h}}}\right) h^{ab} \partial_a X^\mu \partial_b X_\mu + \sqrt{-h}\, \partial_c X^\mu \partial_d X_\mu \\ &= \frac{1}{2\sqrt{-\tilde{h}}^3} \frac{\partial \tilde{h}}{\partial h^{cd}} h^{ab} \partial_a X^\mu \partial_b X_\mu + \sqrt{-h}\, \partial_c X^\mu \partial_d X_\mu \\ &= \frac{1}{2\sqrt{-\tilde{h}}^3} \tilde{h}\, h_{cd}\, h^{ab} \partial_a X^\mu \partial_b X_\mu + \sqrt{-h}\, \partial_c X^\mu \partial_d X_\mu \\ &= \frac{-1}{2(-h)} \sqrt{-h}^3\, h_{cd}\, h^{ab} \partial_a X^\mu \partial_b X_\mu + \sqrt{-h}\, \partial_c X^\mu \partial_d X_\mu \\ &= -\frac{\sqrt{-h}}{2}\, h_{cd}\, h^{ab} \partial_a X^\mu \partial_b X_\mu + \sqrt{-h}\, \partial_c X^\mu \partial_d X_\mu. \end{aligned}$$

From this we derive that the integrands in both actions are in fact identical

$$\begin{aligned} g_{cd} &= \frac{1}{2} h_{cd}\, h^{ab} g_{ab}, \\ \Rightarrow \sqrt{-g} &= \frac{\sqrt{-h}}{2} h^{ab} g_{ab}. \end{aligned}$$

□

### 2.3 Conserved Charges from Lagrangian Symmetries

Let us assume that we are given an action

$$S = \int \prod_{i=0}^{d} \mathrm{d}\xi^i \, \mathscr{L}(\phi^a, \partial_\alpha \phi^a),$$

for fields $\phi^a(\xi^\beta)$ in a $D$ dimensional spacetime and an infinitesimal variation of the fields under which the action remains invariant. We receive the EULER-LAGRANGE-equations for fields by the principle of least action: If the fields $\phi^a$ satisfy the equations of motions the variation of the action with respect to the variation of the fields should vanish, i.e. $\frac{\delta S}{\delta \phi^a} = 0$. First, we derive the total variation of the action as[1]

$$\delta S = \int \mathrm{dvol} \left( \frac{\partial \mathscr{L}}{\partial \phi^a} \delta \phi^a + \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \phi^a)} \delta(\partial_\alpha \phi^a) \right) \tag{2a}$$

$$= \int \mathrm{dvol} \left( \frac{\partial \mathscr{L}}{\partial \phi^a} - \partial_\alpha \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \phi^a)} \right) \delta \phi^a, \tag{2b}$$

where we used that taking the variation of $\phi^a$ commutes with the derivatives with respect to the coordinates and assumed that the fields vanish at all infinities in order for the term arising from partial integration to be zero. The before mentioned condition yields

$$0 \stackrel{!}{=} \frac{\delta S}{\delta \phi^a} = \frac{\partial \mathscr{L}}{\partial \phi^a} - \partial_\alpha \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \phi^a)} \qquad \forall a. \tag{3}$$

If the variation of the fields leaves the Lagrangian itself invariant, we will call it a *symmetry* of the Lagrangian. This is equivalent to the condition[2]

$$0 = \delta \mathscr{L} = \frac{\partial \mathscr{L}}{\partial \phi^a} \delta \phi^a + \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \phi^a)} \delta(\partial_\alpha \phi^a). \tag{4}$$

We will assume that the variation of the fields depends on some parameter $\lambda \in \mathbb{R}$ where the derivative $\frac{\delta \mathscr{L}}{\delta \lambda}$ exists and vanishes, namely, that the variation is *continuous*. We then find with eq. (4)

$$0 = \frac{\delta \mathscr{L}}{\delta \lambda} = \left[ \frac{\partial \mathscr{L}}{\partial \phi^a} - \partial_\alpha \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \phi^a)} \right] \frac{\delta \phi^a}{\delta \lambda} + \partial_\alpha \left[ \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \phi^a)} \frac{\delta \phi^a}{\delta \lambda} \right].$$

The first term vanishes because of eq. (3) and we define the *Noether current* as

$$j^\alpha := \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \phi^a)} \frac{\delta \phi^a}{\delta \lambda} \tag{5}$$

with $\partial_\alpha j^\alpha = 0$. This current is called conserved, since it induces a conserved charge

$$Q = \int \prod_{i=1}^{d} \mathrm{d}\xi^i \, j^0,$$

which we can verify by calculating

$$\frac{\partial Q}{\partial t} = \int \prod_{i=1}^{d} \mathrm{d}\xi^i \, \frac{\partial j^0}{\partial t}$$

$$= - \int \prod_{i=1}^{d} \mathrm{d}\xi^i \, \partial_i j^i = 0,$$

[1] We use for the sake of clarity $\mathrm{dvol} = \prod_{i=0}^{d} \mathrm{d}\xi^i$.

[2] Note that we assumed that the Lagrangian is dependent only on the fields and its first derivatives but not on the coordinates of spacetime.

where the summation was taken over the spatial coordinates and we assumed that the current vanishes at all infinities.

Now let us go back to our two-dimensional world-sheet. We define the *momentum densities* as

$$\mathscr{P}_\mu^\alpha := \frac{\partial \mathscr{L}}{\partial(\partial_\alpha X^\mu)}$$

or more explicitly

$$\mathscr{P}^{\sigma\mu} = -T_0 \frac{\left(X' \cdot \dot{X}\right) \dot{X}^\mu - \dot{X}^2 X^{\mu\,\prime}}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \dot{X}^2 X'^2}}, \tag{6a}$$

$$\mathscr{P}^{\tau\mu} = -T_0 \frac{\left(X' \cdot \dot{X}\right) X^{\mu\,\prime} - X'^2 \dot{X}^\mu}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \dot{X}^2 X'^2}}. \tag{6b}$$

By considering infinitesimal translations $\delta X^\mu = \epsilon^\mu$ where $\epsilon^\mu$ are constant we find that its Noether currents are given by $\mathscr{P}_\mu^\alpha$ and conclusively $\partial_\alpha \mathscr{P}_\mu^\alpha = 0$. The corresponding charge is called *spacetime momentum* and is given by

$$p_\mu(\tau) := \int_0^{\sigma_1} \mathrm{d}\sigma \; \mathscr{P}_\mu^\tau(\tau, \sigma).$$

Since the space now is bounded, it is not at all trivial that $p_\mu(\tau)$ is in fact conserved. We will come back to this in the next section.

## 2.4 Boundary Conditions

In equation eq. (2) we assumed that the quantity

$$\int \mathrm{dvol}\, \partial_\alpha \left( \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \phi^a)} \delta\phi^a \right)$$

vanishes. If we formulate this expression for our world-sheet, we receive the condition

$$\begin{aligned} 0 &= \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma \; \partial_\alpha \left( \frac{\partial \mathscr{L}}{\partial(\partial_\alpha \phi^a)} \delta\phi^a \right) \\ &= \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma \; \partial_\alpha \left( \mathscr{P}_\mu^\alpha \delta X^\mu \right) \\ &= \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int_0^{\sigma_1} \mathrm{d}\sigma \; \left( \partial_\tau \left( \mathscr{P}_\mu^\tau \delta X^\mu \right) + \partial_\sigma \left( \mathscr{P}_\mu^\sigma \delta X^\mu \right) \right). \end{aligned}$$

We will assume that the variation fulfils

$$\delta X^\mu(\tau_i, \sigma) = \delta X^\mu(\tau_f, \sigma) = 0 \quad \forall \sigma \in [0, \sigma_1], \mu,$$

and therefore the integration over the $\tau$-derivative vanishes. We remain with the problem

$$\begin{aligned} 0 &= \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \; \left[ \mathscr{P}_\mu^\sigma \delta X^\mu \right]_0^{\sigma_1} \\ &= \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \; \Big( \mathscr{P}_0^\sigma(\tau, \sigma_1) \delta X^0(\tau, \sigma_1) - \mathscr{P}_0^\sigma(\tau, 0) \delta X^0(\tau, 0) \\ &\qquad + \cdots + \mathscr{P}_d^\sigma(\tau, \sigma_1) \delta X^d(\tau, \sigma_1) - \mathscr{P}_d^\sigma(\tau, 0) \delta X^d(\tau, 0) \Big). \end{aligned}$$

For closed string, where $\sigma = \sigma_1$ is associated with $\sigma = 0$ this relation holds automatically. For open strings on the other hand we derive two possible boundary conditions for the endpoints $\sigma^* \in \{0, \sigma_1\}$:

(i) DIRICHLET boundary condition: $\frac{\partial X^\mu}{\partial \tau}(\tau, \sigma^*) = 0$,

(ii) VON-NEUMANN boundary condition: $\mathscr{P}^\sigma_\mu(\tau, \sigma^*) = 0$.

The DIRICHLET boundary condition corresponds to fixed endpoints, whereas VON-NEUMANN boundary condition represents free-endpoints. The objects to which the endpoints of the strings are connected are called *D-branes*. Thus, the first condition corresponds to a zero-dimensional D-brane, whereas the second one can be realised by a space filling D-brane. In the following we will only be using the free-endpoint boundary condition.

Coming back to section 2.3, we now find that the spacetime momentum $p_\mu$ is indeed conserved for open strings with free-endpoints and closed strings

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} = \int_0^{\sigma_1} \mathrm{d}\sigma\, \partial_\tau \mathscr{P}^\tau_\mu = -\int_0^{\sigma_1} \mathrm{d}\sigma\, \partial_\sigma \mathscr{P}^\sigma_\mu = -\mathscr{P}^\sigma_\mu(\tau, \sigma)|_0^{\sigma_1} = 0.$$

For open strings with fixed endpoints the spacetime momentum fails to be conserved. Nevertheless, one can show that the total momentum of the string *and* the D-brane is conserved.

## 2.5 The complete Momentum Current

In section 2.3 we found that $\mathscr{P}^\alpha_\mu$ is a conserved current and concluded that

$$p_\mu(\tau) = \int_0^{\sigma_1} \mathrm{d}\sigma\, \mathscr{P}^\tau_\mu$$

is conserved, i.e. $\frac{\mathrm{d}}{\mathrm{d}\tau} p_\mu = 0$ for closed strings and open string with free endpoints. We integrated over a constant value of $\tau$, thus it is not trivial that spacetime momentum is independent of the gauge we choose. As we will show in the following, the integral is the same for any path along the world-sheet with endpoints $\{0, \sigma_1\}$.

*Proof.* Consider the flux $F$ of $\mathscr{P}^\alpha_\mu$ out of some region $R$ on the world-sheet of an open string with free endpoints

$$F = \left(\mathscr{P}^\tau_\mu, \mathscr{P}^\sigma_\mu\right) \cdot (\mathrm{d}\sigma, -\mathrm{d}\tau) = \mathscr{P}^\tau_\mu \mathrm{d}\sigma - \mathscr{P}^\sigma_\mu \mathrm{d}\tau.$$

The outgoing flux over some path $\Gamma = \partial R$ is given by

$$\begin{aligned} P(\Gamma) &:= \oint_\Gamma \left(\mathscr{P}^\tau_\mu \mathrm{d}\sigma - \mathscr{P}^\sigma_\mu \mathrm{d}\tau\right) \\ &= \int_R \mathrm{d}\left(P^\tau_\mu \mathrm{d}\sigma - \mathscr{P}^\sigma_\mu \mathrm{d}\tau\right) \\ &= \int_R \mathrm{d}\sigma\, \mathrm{d}\tau \left(P^\tau_\mu + \mathscr{P}^\sigma_\mu\right) = 0, \end{aligned}$$

where we used STOKE's theorem, the antisymmetry of the exterior product and the fact that $\mathscr{P}^\alpha_\mu$ is a conserved current. Now let $\Gamma = \bar{\gamma} + \beta + \gamma + \alpha$ where $\gamma$ and $\bar{\gamma}$ are paths connecting both ends of the string and $\alpha$ and $\beta$ are the paths connecting $\gamma$ and $\bar{\gamma}$ at $\sigma = 0$ resp. $\sigma = \sigma_1$. We choose $\bar{\gamma}$ to be a path of constant $\tau$ and find

$$P(\Gamma) = \left(\int_{\bar{\gamma}} + \int_\beta + \int_\gamma + \int_\alpha\right)\left(\mathscr{P}^\tau_\mu \mathrm{d}\sigma - \mathscr{P}^\sigma_\mu \mathrm{d}\tau\right) = 0.$$

Note that for the integrals over $\alpha$ and $\beta$ the second term vanishes due to the constance of $\tau$. Since the endpoints satisfy the NEUMANN-boundary condition $\mathscr{P}^\tau_\mu = 0$ the first term vanishes as well. Since we chose the orientation of $\gamma$ in the opposite direction as for $\bar{\gamma}$ the statement is proven. The proof for closed strings is analogous, where $\gamma$ and $\bar{\gamma}$ are winding once around the world-sheet. □

We redefine the spacetime momentum of the string as

$$p_\mu = \int_\gamma \left( \mathscr{P}^\tau_\mu \mathrm{d}\sigma - \mathscr{P}^\tau_\mu \mathrm{d}\sigma \right),$$

where $\gamma$ fulfils the conditions mentioned above.

## 2.6 Lorentz Symmetry and associated Currents

Every LORENTZ transformation is linear and leaves the quadratic form $\eta_{\mu\nu} X^\mu X^\nu$ invariant. Thus, we examine the transformation

$$X^\mu \to X^\mu + \delta X^\mu, \quad \delta X^\mu = \epsilon^{\mu\nu} X_\nu$$

where $\epsilon$ is a tensor of infinitesimal norm. To ensure that the quadratic form is left invariant under the variation we require

$$0 \overset{!}{=} \delta \left( \eta_{\mu\nu} X^\mu X^\nu \right) = 2 \eta_{\mu\nu} X^\mu \delta X^\nu = 2 \epsilon^{\nu\rho} X_\nu X_\rho.$$

After a decomposition of $\epsilon^{\mu\nu}$ into a symmetric and a antisymmetric term we find that the symmetric term must vanish to satisfy the relation above and conclude that[3]

$$\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}.$$

Since every term in the Lagrangian of the NAMBO-GOTO-action in eq. (1) is of the form $\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^1} \frac{\partial X^\mu}{\partial \xi^2}$ for $\xi^i \in \{\tau, \sigma\}$ it suffices to show that this expression is invariant under the variation $\delta X^\mu$ to conclude that it is a (continuous) symmetry of the Lagrangian. We verify

$$\begin{aligned}
\delta \left( \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^1} \frac{\partial X^\nu}{\partial \xi^2} \right) &= \eta_{\mu\nu} \left( \frac{\partial \delta X^\mu}{\partial \xi^1} \frac{\partial X^\nu}{\partial \xi^2} + \frac{\partial X^\mu}{\partial \xi^1} \frac{\partial \delta X^\nu}{\partial \xi^2} \right) \\
&= \eta_{\mu\nu} \left( \epsilon^{\mu\sigma} \frac{\partial X_\sigma}{\partial \xi^1} \frac{\partial X^\nu}{\partial \xi^2} + \epsilon^{\nu\rho} \frac{\partial X^\mu}{\partial \xi^1} \frac{\partial X_\rho}{\partial \xi^2} \right) \\
&= \epsilon^{\mu\sigma} \frac{\partial X_\sigma}{\partial \xi^1} \frac{\partial X_\mu}{\partial \xi^2} + \epsilon^{\nu\rho} \frac{\partial X_\nu}{\partial \xi^1} \frac{\partial X_\rho}{\partial \xi^2} \\
&= \underbrace{\left( \epsilon^{\rho\nu} + \epsilon^{\nu\rho} \right)}_{=0} \frac{\partial X_\nu}{\partial \xi^1} \frac{\partial X_\rho}{\partial \xi^2} = 0.
\end{aligned}$$

It follows with eq. (5) that the associated current of this symmetry is given by

$$j^\alpha_{\mu\nu} = \frac{\partial \mathscr{L}}{\partial(\partial_\alpha X^\rho)} \frac{\delta X^\rho}{\delta \epsilon^{\mu\nu}} = \mathscr{P}^\alpha_\mu X_\nu.$$

Since $\epsilon^{\mu\nu}$ is antisymmetric we can rewrite the variation as $\delta X^\mu = \frac{1}{2} \left( \epsilon^{\mu\nu} - \epsilon^{\nu\mu} \right) X_\nu$. This allows us to find antisymmetric conserved currents

$$M^\alpha_{\mu\nu} = X_\mu \mathscr{P}^\alpha_\nu - X_\nu \mathscr{P}^\alpha_\mu,$$

where we used the equivalent transformation and switched the total sign. The conserved charge then is given by

$$M_{\mu\nu} = \int_\gamma \left( M^\tau_{\mu\nu} \mathrm{d}\sigma - M^\sigma_{\mu\nu} \mathrm{d}\tau \right).$$

In exactly the same way as for the spacetime momentum from the last section we find that this expression for an arbitrary path $\gamma$ connecting both ends of the string resp. winding once around the world-sheet is well-defined.

[3] The LIE algebra of the LORENTZ group is given by $\mathfrak{so}(1,d) = \left\{ \Lambda \in M^{n \times n}(\mathbb{R}) \,|\, \eta \Lambda = \Lambda^{\mathrm{T}} \eta \right\}$. Since the elements of the LIE algebra represent the components of a tensor of type (1,1) they fulfil $\eta^{\mu\nu} \Lambda_\nu^{\ \rho} = -\Lambda_\nu^{\ \rho} \eta^{\nu\mu} \Leftrightarrow \Lambda^{\mu\nu} = -\Lambda^{\nu\mu}$.

## 3 Light Cone relativistic Strings

### 3.1 Parameterisations and Gauges

In the following we are going to examine parameterisations for $\tau$ of the form

$$n_\mu X^\mu(\tau,\sigma) = \lambda\tau$$

where $n_\mu \in \mathbb{R}^D$ and $\lambda \in \mathbb{R}$. To ensure that two points on the string (with constant $\tau$) are connected via a spacelike vector, the vector $n_\mu$ must be timelike. Note that a gauge defined in that manner is not Lorentz covariant, since we interpret $n_\mu X^\mu$ as a choice of linear combination of coordinates and not as a dot product between two Lorentz covariant vectors. To factor out the length of $n_\mu$, we redefine the gauge as

$$n\cdot X = \tilde{\lambda}\,(n\cdot p)\,\tau. \tag{7}$$

If the D-branes – to which the string endpoints are connected – are not space-filling, it may happen that not every component of the spacetime momentum is conserved. Nevertheless, we will assume that it is still possible to find a vector $n_\mu$ such that the linear combination of its combination remains constant.

For convenience we will introduce a new constant[4]

$$\alpha' := \frac{1}{2\pi\, T_0},$$

where $T_0$ is the string tension introduced in the definition of the Nambo-Goto-action in eq. (1). We will take $\tilde{\lambda} = \beta\,\alpha'$ where $\beta = 1$ for closed strings and $\beta = 2$ for open strings, yielding

$$n\cdot X = \beta\,\alpha'\,(n\cdot p)\,\tau. \tag{8}$$

The momentum density $\mathcal{P}^{\tau\mu}$ transforms according to $\mathcal{P}^{\tau\mu}(\tau,\tilde{\sigma}) = \frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\sigma}\mathcal{P}^{\tau\mu}(\tau,\sigma)$. Heuristically, this transformation behaviour can be obtained by taking into account that $\mathcal{P}^{\tau\mu}$ contains two $\sigma$-derivatives in the numerator while the denominator only has one and making use of Leibniz's notation to write $\frac{\partial X^\mu}{\partial\sigma} = \frac{\partial X^\mu}{\partial\tilde{\sigma}}\frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\sigma}$. This tells us that we can choose a $\sigma$-parameterisation $\tilde{\sigma}$, such that

$$n\cdot\mathcal{P}^\tau(\tau,\sigma) = \frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\sigma}\,n\cdot\mathcal{P}^\tau(\tau,\tilde{\sigma}) = a(\tau)$$

is independent of $\sigma$. By rescaling the parameterisation we can choose $\sigma \in [0,\frac{2\pi}{\beta}]$ and derive that $a(\tau)$ is in fact independent of $\tau$

$$\frac{2\pi}{\beta}\,a(t) = \int_0^{\frac{2\pi}{\beta}} \mathrm{d}\sigma\, n\cdot\mathcal{P}^\tau(\tau,\sigma) = n\cdot p.$$

We conclude that

$$n\cdot\mathcal{P}^\tau = \beta\,\frac{n\cdot p}{2\pi} \tag{9}$$

is a world-sheet constant.

We will from now on assume that for open strings $n\cdot\mathcal{P}^\tau = 0$ at the endpoints. This is motivated by the idea that it implies the conservation of $n\cdot p$. Since $\partial_\alpha\mathcal{P}^\alpha_\mu = 0$ we find with eq. (9)

$$\frac{\partial}{\partial\sigma}\left(n\cdot\mathcal{P}^\sigma\right) = 0 \tag{10}$$

and therefore that $n\cdot\mathcal{P}^\sigma = 0$ along the open string. For closed strings eq. (10) is fulfilled as well, but it is not trivial that $n\cdot\mathcal{P}^\sigma$ vanishes for all points. To prove this claim we select an arbitrary point on the world-sheet and denote it with $\sigma = 0$. We construct the rest of the $\sigma = 0$ line by requiring its tangent points to be orthogonal to $X^{\mu\prime}$ yielding $\dot{X}\cdot X' = 0$ along the line. With eq. (6a) and eq. (8) it now follows that $n\cdot\mathcal{P}^\sigma$ vanishes for $\sigma = 0$ and thus for the whole string.

[4] The quantity $\alpha'$ is the fraction between the total angular momentum divided by the energy squared of a rigidly rotating open string. The derivation would go beyond the scope of this report but can be looked up in [1] on page 168.

### 3.2 Wave Equation

In the last section we derived that $n \cdot \mathscr{P}^\sigma = 0$ and with eq. (8) we now find

$$0 = n \cdot \mathscr{P}^\sigma = -\frac{1}{2\pi\alpha'} \frac{\left(\dot{X} \cdot X'\right) \partial_\tau (n \cdot X)}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \dot{X}^2 X'^2}}.$$

With the assumption that $n \cdot p \neq 0$ this implies

$$\dot{X} \cdot X' = 0. \tag{11}$$

Together with eq. (9) this constraint yields

$$\frac{\beta}{2\pi} n \cdot p = n \cdot \mathscr{P}^\tau = \frac{1}{2\pi\alpha'} \frac{X'^2 \partial_\tau (n \cdot X)}{\sqrt{-\dot{X}^2 X'^2}} = \frac{1}{2\pi\alpha'} \frac{X'^2 \beta \alpha' n \cdot p}{\sqrt{-\dot{X}^2 X'^2}}$$

and after dividing both sides by $\frac{\beta n \cdot p}{2\pi}$ and requiring $X'^2 \neq 0$ we find

$$\dot{X}^2 + X'^2 = 0. \tag{12}$$

We can rewrite the two constraints in eq. (11) and eq. (12) into one equation

$$\left(\dot{X} \pm X'\right)^2 = 0. \tag{13}$$

We now want to simplify the expression for the momentum densities $\mathscr{P}^{\alpha\mu}$. Since we chose $n_\mu$ in a way that all points with the same value for $\tau$ are connected via a spacelike vector, $X^{\mu\prime}$ is also spacelike, i.e. $X'^2 > 0$ and we find

$$\sqrt{-\dot{X}^2 X'^2} = \sqrt{X'^2 X'^2} = X'^2.$$

Plugging this into the formula for $\mathscr{P}^{\alpha\mu}$ we find

$$P^{\tau\mu} = \frac{1}{2\pi\alpha} \dot{X}^\mu, \tag{14a}$$

$$P^{\sigma\mu} = -\frac{1}{2\pi\alpha} X^{\mu\prime} \tag{14b}$$

and with the conservation of the current $\partial_\alpha \mathscr{P}^{\alpha\mu} = 0$

$$\ddot{X}^\mu - X^{\mu\prime\prime} = 0.$$

This is an ordinary wave equation for the mapping functions $X^\mu$. Thus, the general solution for an open string with free endpoints $X'^\mu(\tau, \{0, \pi\}) \propto \mathscr{P}^{\sigma\mu}(\tau, \{0, \pi\}) = 0$ is given by

$$X^\mu(\tau, \sigma) = x_0^\mu + \sqrt{2\alpha'} \alpha_0^\mu \tau - \imath \sqrt{2\alpha'} \sum_{n=1}^{\infty} \left(a_n^{\mu *} e^{\imath n \tau} - a_n^\mu e^{-\imath n \tau}\right) \frac{\cos(n\sigma)}{\sqrt{n}}, \tag{15}$$

where $a_n^\mu$ are complex coefficients. Since the integrals over cosines of the form $\int_0^\pi \mathrm{d}x \cos(nx)$ vanish, we find

$$p_\mu = \int_0^\pi \mathrm{d}\sigma \, \mathscr{P}^{\tau\mu} = \int_0^\pi \mathrm{d}\sigma \frac{1}{2\pi\alpha'} \dot{X}^\mu = \frac{1}{2\pi\alpha'} \pi \sqrt{2\alpha'} \alpha_0^\mu$$

and therefore

$$\alpha_0^\mu = \sqrt{2\alpha'} p^\mu.$$

This result seems natural, since one would expect a string without any oscillations to move with a velocity proportional to its momentum. We define for all $n \in \mathbb{N}$

$$\alpha_n^\mu := \sqrt{n} a_n^\mu, \qquad \alpha_{-n}^\mu := \sqrt{n} a_n^{\mu *}. \tag{16}$$

Inserting the newly defined $\alpha_n^\mu$ into eq. (15) yields

$$X^\mu(\tau,\sigma) = x_0 + \sqrt{2\alpha'}\alpha_0^\mu\tau + \imath\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-\imath n\tau}\cos(n\sigma).$$

Furthermore, we find the derivatives with respect to $\tau$ and $\sigma$

$$\dot{X}^\mu(\tau,\sigma) = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^\mu\cos(n\sigma)e^{-\imath n\tau}, \tag{17a}$$

$$X^{\mu\prime}(\tau,\sigma) = -\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^\mu\sin(n\sigma)e^{-\imath n\tau}. \tag{17b}$$

and thus

$$\dot{X}^\mu \pm X^{\mu\prime} = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-\imath n(\tau\pm\sigma)}. \tag{18}$$

In the following section we will introduce the light cone gauge, a special case of eq. (7) and derive restrictions on the coefficients $\alpha_n^\mu$ with eq. (13) and eq. (18).

### 3.3 Light Cone Solution of Equations of Motion

The light cone coordinates of a vector $\mathbf{v}^\mu$ are given by

$$\mathbf{v}^\pm = \frac{1}{\sqrt{2}}\left(\mathbf{v}^0 \pm \mathbf{v}^1\right),$$

where the remaining $D-2$ coordinates remain the same. The light cone gauge then is taking $n_\mu = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \mathbf{0}\right)$ in eq. (8), yielding

$$X^+ = \beta\alpha' p^+\tau, \qquad p^+ = \frac{2\pi}{\beta}\mathcal{P}^{\tau+}.$$

We now want to examine the restriction eq. (13) in this gauge. The scalar-product translates into

$$a\cdot b = -a^+b^- - a^-b^+ + \mathbf{a}_T\cdot\mathbf{b}_T,$$

as one readily verifies. Here we introduced the *transverse* components $\mathbf{a}_T$ of a vector $\mathbf{a}$, representing the remaining $D-2$ components. In light cone coordinates, the restriction eq. (13) is given by

$$0 = \left(\dot{X}\pm X'\right)^2 = -2\left(\dot{X}^+ \pm X^{+\prime}\right)\left(\dot{X}^- \pm X^{-\prime}\right) + \left(\dot{X}^I \pm X^{I\prime}\right)^2,$$

where $I = (2,\ldots,D)$. With $X^{+\prime} = 0$ we find[5]

$$\dot{X}^- \pm X^{-\prime} = \frac{1}{\beta\alpha'}\frac{1}{2p^+}\left(\dot{X}^I \pm X^{I\prime}\right)^2. \tag{19}$$

We note that this equation determines both $X^{-\prime}$ and $\dot{X}^-$ in terms of the derivatives of the transverse coordinates. Upon a constant of integration we call $x_0^-$, the movement of the string in the $X^-$ component is a function of the transverse coordinates and we find, that the whole string is described by the variables

$$X^I(\tau,\sigma),\ p^+,\ x_0^-.$$

Let us now consider an open string, i.e. taking $\beta = 2$. The general solution for the transverse directions is given by

$$X^I(\tau,\sigma) = x_0^I + \sqrt{2\alpha'}\alpha_0^I\tau + \imath\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^I e^{-\imath n\tau}\cos(n\sigma).$$

[5] We will always assume that $p^+ > 0$. Although $p^+$ naturally is non-negative, a particle with the speed of light moving into the negative $x^1$ direction would yield $p^+ = 0$.

For the + component we get

$$X^+(\tau,\sigma) = 2\alpha' p^+ \tau = \sqrt{2\alpha'}\,\alpha_0^+ \tau$$

from which we identify $x_0^+ = 0$ and $\alpha_n^+ = \alpha_{-n}^+ = 0\ \forall n \in \mathbb{N}$. With eq. (18) for both the − component and the transverse components and eq. (19) we derive an explicit formula for the coefficients $\alpha_n^-$ in terms of the transverse ones

$$\begin{aligned}\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^- e^{-\imath n(\tau\pm\sigma)} &= \frac{1}{2p^+}\sum_{p,q\in Z}\alpha_p^I\alpha_q^I e^{-\imath(p+q)(\tau\pm\sigma)}\\ &= \frac{1}{2p^+}\sum_{p,n\in\mathbb{Z}}\alpha_p^I\alpha_{n-p}^I e^{-\imath n(\tau\pm\sigma)}\\ &= \frac{1}{2p^+}\sum_{n\in\mathbb{Z}}\left(\sum_{p\in\mathbb{Z}}\alpha_p^I\alpha_{n-p}^I\right)e^{-\imath n(\tau\pm\sigma)},\end{aligned}$$

where we find via a comparison of coefficients and the transverse Virasoro modes $L_n^\perp$

$$\sqrt{2\alpha'}\,\alpha_n^- = \frac{1}{p^+}L_n^\perp, \qquad L_n^\perp := \frac{1}{2}\sum_{p\in\mathbb{Z}}\alpha_{n-p}^I\alpha_n^I.$$

From our previous definitions it follows that

$$\sqrt{2\alpha'}\,\alpha_0^- = 2\alpha' p^- = \frac{1}{p^+}L_0^\perp \quad\Rightarrow\quad L_0^\perp = 2\alpha' p^+ p^-.$$

# 4 Relativistic quantum open String

## 4.1 Light Cone Hamiltonian and Commutators

In the last section we found that the momentum densities are proportional to the respective derivatives of the mapping functions $X^\mu$. This relation holds for the light cone gauge as well, since it is only a special case with $n_\mu = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \mathbf{0}\right)$ of the gauge in which we derived the result. Solving eq. (19) for $\dot{X}^-$ we find

$$\dot{X}^- = \frac{1}{2\alpha'}\frac{1}{2p^+}\left(\dot{X}^I\dot{X}^I + X^{I\prime}X^{I\prime}\right)$$

and thus

$$\mathscr{P}^{\tau-} = \frac{1}{2\pi\alpha'}\dot{X}^- = \frac{\pi}{2p^+}\left(\mathscr{P}^{\tau I}\mathscr{P}^{\tau I} + \frac{X^{I\prime}X^{I\prime}}{(2\pi\alpha')^2}\right).$$

To quantise the string, we choose the set of $\tau$ independent Schrödinger operators describing the string as

$$X^I(\sigma),\ x_0^-,\ \mathscr{P}^{\tau I}(\sigma)\ \&\ p^+.$$

The corresponding Heisenberg operators then are only implicitly $\tau$ dependent. We postulate

$$\left[X^I(\sigma), \mathscr{P}^{\tau J}(\sigma')\right] = \imath\eta^{IJ}\delta(\sigma-\sigma'), \tag{20a}$$

$$\left[X^I(\sigma), X^J(\sigma')\right] = \left[\mathscr{P}^{\tau I}(\sigma), \mathscr{P}^{\tau J}(\sigma')\right] = 0, \tag{20b}$$

$$\left[x_0^-, p^+\right] = -\imath, \tag{20c}$$

while $x_0^-$ and $p^+$ commute with both $X^I$ and $\mathscr{P}^{\tau I}$. Since all commutators commute with the time propagator, the Heisenberg operators satisfy the same relations. With $\partial_\tau = 2\alpha' p^+\partial_+$ and $\partial_+ \leftrightarrow -p_+ = p^-$ we expect the

HAMILTONian of the system to be[6]

$$H(\tau) = \partial_\tau = 2\alpha' p^- p^+ = 2\alpha' p^+ \int_0^\pi \mathrm{d}\sigma \mathscr{P}^{\tau -} \tag{21}$$

$$= \pi \alpha' \int_0^\pi \mathrm{d}\sigma \left( \mathscr{P}^{\tau I} \mathscr{P}^{\tau I} + \frac{X^{I\prime} X^{I\prime}}{(2\pi\alpha')^2} \right) \tag{22}$$

$$= L_0^\perp. \tag{23}$$

Since $H(\tau)$ does not have an explicit $\tau$ dependence, it follows that $H(\tau) \equiv H$ is constant. Furthermore, we immediately see that $[H, x_0^-(\tau)] = [H, p^+] = 0$, implying that $x_0^-(\tau) \equiv x_0^-$ and $p^+(\tau) \equiv p^+$ are indeed constant. To verify the equations of motion for $X^I(\tau,\sigma)$ we calculate

$$\imath \dot{X}^I(\tau,\sigma) = [X^I(\tau,\sigma), H(\tau)] = \left[ X^I(\tau,\sigma), \pi\alpha' \int_0^\pi \mathrm{d}\sigma' \mathscr{P}^{\tau J}(\tau,\sigma') \mathscr{P}^{\tau J}(\tau,\sigma') \right] \tag{24}$$

$$= \pi\alpha' \int_0^\pi \mathrm{d}\sigma' \left[ X^I(\tau,\sigma), \left(\mathscr{P}^{\tau J}(\tau,\sigma')\right)^2 \right] \tag{25}$$

$$= 2\pi\alpha' \imath \mathscr{P}^{\tau.\sigma}. \tag{26}$$

The time evolution of $\mathscr{P}^{\tau I}(\tau,\sigma)$ in the HEISENBERG picture yields the wave equation via

$$\begin{aligned}
\frac{\imath}{2\pi\alpha'} \ddot{X}^I(\tau,\sigma) = \imath \dot{\mathscr{P}}^{\tau I}(\tau,\sigma) &= \left[ \mathscr{P}^{\tau I}(\tau,\sigma), \frac{1}{4\pi\alpha'} \int_0^\pi \mathrm{d}\sigma' X^{J\prime}(\tau,\sigma') X^{J\prime}(\tau,\sigma') \right] \\
&= \frac{1}{4\pi\alpha'} \int_0^\pi \mathrm{d}\sigma' \left( X^{J\prime}(\tau,\sigma') \left[ \mathscr{P}^{\tau I}(\tau,\sigma), X^{J\prime}(\tau,\sigma') \right] + \left[ \mathscr{P}^{\tau I}(\tau,\sigma), X^{J\prime}(\tau,\sigma') \right] X^{J\prime}(\tau,\sigma') \right) \\
&\overset{\text{P.I.}}{=} -\frac{1}{4\pi\alpha'} \int_0^\pi \mathrm{d}\sigma' \left( X^{J\prime\prime}(\tau,\sigma')(-\imath)\eta^{IJ}\delta(\sigma-\sigma') + (-\imath)\eta^{IJ}\delta(\sigma-\sigma') X^{J\prime\prime}(\tau,\sigma') \right) \\
&= \frac{\imath}{2\pi\alpha} X^{I\prime\prime}(\tau,\sigma).
\end{aligned}$$

We differentiate eq. (20a) once with respect to $\sigma$ and find

$$[X^{I\prime}(\tau,\sigma), \dot{X}^J(\tau,\sigma')] = 2\pi\alpha' \imath \eta^{IJ} \frac{\mathrm{d}}{\mathrm{d}\sigma} \delta(\sigma-\sigma')$$

and from eq. (20b) it follows that

$$[X^{I\prime}(\tau,\sigma), X^{J\prime}(\tau,\sigma')] = [\dot{X}^I(\tau,\sigma), \dot{X}^J(\tau,\sigma')] = 0.$$

Combining the last two equation yields

$$\begin{aligned}
\left[ \left(\dot{X}^I \pm_1 X^{I\prime}\right)(\tau,\sigma), \left(\dot{X}^J \pm_2 X^{J\prime}\right)(\tau,\sigma') \right] &= \pm_2 \underbrace{\left[ \dot{X}^I(\tau,\sigma), X^{J\prime}(\tau,\sigma') \right]}_{\substack{= -2\pi\alpha' \imath \eta^{JI} \frac{\mathrm{d}}{\mathrm{d}\sigma'} \delta(\sigma'-\sigma) \\ = +2\pi\alpha' \imath \eta^{IJ} \frac{\mathrm{d}}{\mathrm{d}\sigma} \delta(\sigma-\sigma')}} \pm_1 \underbrace{\left[ X^{I\prime}(\tau,\sigma), \dot{X}^J(\tau,\sigma') \right]}_{= 2\pi\alpha' \imath \eta^{IJ} \frac{\mathrm{d}}{\mathrm{d}\sigma} \delta(\sigma-\sigma')} \\
&= \pm_1 \delta_{\pm_1,\pm_2} 4\pi\alpha' \imath \eta^{IJ} \frac{\mathrm{d}}{\mathrm{d}\sigma} \delta(\sigma-\sigma').
\end{aligned} \tag{27}$$

## 4.2 Commutation Relations for Oscillators

We will now examine the commutation relations for the oscillators $\alpha_n^I$. To do so we will introduce a new operator $A(\tau,\sigma)$ and examine the commutation properties with itself. In section 3.2 we found that the solution to the wave equation with NEUMANN boundary conditions satisfy

$$\left(\dot{X}^I \pm X^{I\prime}\right)(\tau,\sigma) = \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha_n^I e^{-\imath n(\tau\pm\sigma)}, \qquad \sigma \in [0,\pi]$$

[6] In section 4.4 we derive that the operator $L_0^\perp$ and thus the HAMILTONian must be shifted by an additional real number $a$. Since we are considering the commutator with $H$, an additional constant has no effects on the equations of motion of the string.

and thus

$$\left(\dot{X}^I - X^{I\prime}\right)(\tau,\sigma) = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^I e^{-\imath n(\tau+\sigma)}, \qquad \sigma \in [-\pi, 0].$$

We define a new operator

$$A^I(\tau,\sigma) := \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^I e^{-\imath n(\tau+\sigma)}, \tag{28a}$$

$$= \begin{cases} \left(\dot{X}^I + X^{I\prime}\right)(\tau,\sigma) & \sigma\in[0,\pi], \\ \left(\dot{X}^I - X^{I\prime}\right)(\tau,-\sigma) & \sigma \in [-\pi,0]. \end{cases} \tag{28b}$$

which is $2\pi$-periodic in $\tau$ and in $\sigma$. As mentioned before, we are interested in the commutator $\left[A^I(\tau,\sigma),\, A^J(\tau,\sigma')\right]$. With eq. (27) we find for $\sigma \in [-\pi,\pi]$

$$\left[A^I(\tau,\sigma),\, A^J(\tau,\sigma')\right] = 4\pi\,\alpha'\,\imath\eta^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma'). \tag{29}$$

Note that the minus sign occurring when $\sigma\in[-\pi,0]$ cancels out with the sign appearing in the derivative with respect to $\sigma$. On the other hand, we can instead of using eq. (28b) also calculate the commutator of $A^I$ with itself with eq. (28a). This yields with eq. (29)

$$\sum_{m,n\in\mathbb{Z}} e^{-\imath m(\tau+\sigma)}\, e^{-\imath n(\tau+\sigma)}\,\left[\alpha_m^I,\, \alpha_n^J\right] = 2\pi\,\imath\eta^{IJ}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta(\sigma-\sigma').$$

To extract any information out of this equation, we apply on both sides the operator $\int_0^{2\pi}\frac{\mathrm{d}\sigma}{2\pi}e^{\imath m\sigma}\int_0^{2\pi}\frac{\mathrm{d}\sigma'}{2\pi}e^{\imath n\sigma'}$ and find with the orthogonality of $e^{\imath m\sigma}$ under the inner product product $\langle\phi|\psi\rangle = \int_0^{2\pi}\mathrm{dvol}\,\phi\cdot\overline{\psi}$

$$\begin{aligned}\left[\alpha_m^I,\, \alpha_n^J\right] &= e^{\imath(m+n)\tau}\,\imath\eta^{IJ}\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma\, e^{\imath m\sigma}\frac{\mathrm{d}}{\mathrm{d}\sigma}\int_0^{2\pi}\mathrm{d}\sigma'\, e^{\imath n\sigma'}\,\delta(\sigma-\sigma') \\ &= e^{\imath(m+n)\tau}\,\imath\eta^{IJ}\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma\, e^{\imath m\sigma}\frac{\mathrm{d}}{\mathrm{d}\sigma}e^{\imath n\sigma} \\ &= -e^{\imath(m+n)\tau}\,n\eta^{IJ}\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma\, e^{\imath(m+n)\sigma} \\ &= m\eta^{IJ}\delta_{m+n,0}.\end{aligned} \tag{30}$$

It follows that $\alpha_0^I$ commutes with all other oscillators. This was to be expected, since $\alpha_0^I = \sqrt{2\alpha'}\,p^I$ and $p^I$ only has a non-vanishing commutator with $x_0^I$. Now we want to calculate the commutator $\left[x_0^I,\, \alpha_n^J\right]$. In section 3.2 we found that the transverse components $X^I$ of the string are described by

$$X^I(\tau,\sigma) = x_0^I + \sqrt{2\alpha'}\,\alpha_0^I\,\tau + \imath\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n}\alpha_n^I\cos(n\sigma)\,e^{-\imath n\tau}.$$

We find with eq. (20a) and $\int_0^\pi \mathrm{d}\sigma\cos(n\sigma) = 0$

$$\begin{aligned}2\alpha'\,\imath\eta^{IJ} &= \frac{1}{\pi}\int_0^\pi \mathrm{d}\sigma\,\left[X^I(\tau,\sigma),\, \dot{X}^I(\tau,\sigma')\right] \\ &= \left[x_0^I + \sqrt{2\alpha'}\,\alpha_0^I\tau,\, \dot{X}^I(\tau,\sigma)\right] \\ &= \left[x_0^I,\, \dot{X}^I(\tau,\sigma)\right] \\ &= \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\left[x_0^I,\, \alpha_n^J\right]\cos(n\sigma)\,e^{-\imath n\tau} \\ &= \sqrt{2\alpha'}\left[x_0^I,\, \alpha_0^J\right] + \sqrt{2\alpha'}\sum_{n=1}^{\infty}\left[x_0^I,\, \alpha_n^J e^{-\imath n\tau} + \alpha_{-n}^J e^{\imath n\tau}\right]\cos(n\sigma).\end{aligned}$$

Applying the operator $\frac{1}{\pi}\int_0^\pi \mathrm{d}\sigma' \cos(n\sigma')$ to both sides gives us

$$[x_0^I, \alpha_n^I e^{-\imath n\tau} + \alpha_{-n}^J e^{\imath n\tau}] = 0.$$

Since this relation has to hold for all values of $\tau$, we find that $x_0^I$ commutes with all oscillators $\alpha_n^J$ with $n \neq 0$

$$[x_0^I, \alpha_n^J] = 0, \qquad \forall n \in \mathbb{Z}\setminus\{0\}$$

and furthermore

$$\left[x_0^I, \alpha_0^J\right] = \sqrt{2\alpha'}\,\imath\eta^{IJ}. \tag{31}$$

The commutation relation eq. (30) simplify for the $a_n^I$ defined in eq. (16) to[7]

$$\left[a_m^I, a_n^{J\dagger}\right] = \delta_{m,n}\eta^{IJ}, \tag{32a}$$

$$[a_m^I, a_n^J] = \left[a_m^{I\dagger}, a_n^{J\dagger}\right] = 0. \tag{32b}$$

We conclude with the theory of the quantum harmonic oscillator that $a_n^I$ resp. $a_n^{I\dagger}$ are annihilation resp. creation operators of the open string with free endpoints. Furthermore, we have seen that the full set of operators describing the string is given by four zero modes $(x_0^I, x_0^-, p^I, p^+)$ and an infinite set of creation and annihilation operators.

### 4.3 Transverse Virasoro Operators

We now are interested in the mode expansion of $X^+(\tau,\sigma)$ and $X^-(\tau,\sigma)$. It was found in section 3.3 that

$$X^+(\tau,\sigma) = 2\alpha' p^+ \tau = \sqrt{2\alpha'}\,\alpha_0^+ \tau, \qquad x_0^+ = 0,\ \alpha_n^+ = 0\,\forall n \neq 0,$$
$$X^-(\tau,\sigma) = x_0^- + \sqrt{2\alpha'}\,\alpha_0^- \tau + \imath\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}\setminus\{0\}} \alpha_n^- e^{-\imath n\tau} \frac{\cos(n\sigma)}{n},$$

where

$$\sqrt{2\alpha'}\,\alpha_n^- = \frac{1}{p^+} L_n^\perp, \qquad L_n^\perp \equiv \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_{n-p}^I \alpha_p^I. \tag{33}$$

The Virasoro modes now become the Virasoro operators. To fulfil this transformation we have to apply the correct ordering inside the sum of $L_n^\perp$. To guarantee that the expectation value of $L_n^\perp$ on the vacuum state is zero, we require all annihilation resp. creation operators to stand on the right resp. left. We rewrite $L_0^\perp$ as

$$L_0^\perp = \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_{-p}^I \alpha_p^I = \frac{1}{2}\alpha_0^I\alpha_0^I + \frac{1}{2}\sum_{p=1}^\infty \alpha_{-p}^I\alpha_p^I + \frac{1}{2}\sum_{p=1}^\infty \alpha_p^I \alpha_{-p}^I.$$

While the first two terms are already normal ordered, the last one needs special consideration:

$$\begin{aligned}\frac{1}{2}\sum_{p=1}^\infty \alpha_p^I\alpha_{-p}^I &= \frac{1}{2}\sum_{p=1}^\infty \left(\alpha_{-p}^I\alpha_p^I + \left[\alpha_p^I, \alpha_{-p}^I\right]\right)\\ &= \frac{1}{2}\sum_{p=1}^\infty \alpha_{-p}^I\alpha_p^I + \frac{1}{2}\sum_{p=1}^\infty p\eta^{II}\\ &= \frac{1}{2}\sum_{p=1}^\infty \alpha_{-p}^I\alpha_p^I + \frac{D-2}{2}\sum_{p=1}^\infty p.\end{aligned}$$

[7] By going from the classic theory to the theory of quantum mechanics we replace the complex conjugation $^*$ by the $^\dagger$ symbol.

Thus, normal ordering the operator $L_0^\perp$ yields a seemingly infinite series. In appendix A.1 we derive

$$\zeta(-1) = \sum_{n=1}^{\infty} n = -\frac{1}{12}$$

as the limit of the analytical continuation of the zeta-function. We *redefine* the operator $L_0^\perp$ as

$$L_0^\perp := \frac{1}{2}\sum_{p=1}^{\infty} :\alpha_{-p}^I \alpha_p^I: - a, \quad \text{with} \quad a := \frac{D-2}{2}\sum_{p=1}^{\infty} p = -\frac{D-2}{24}. \tag{34}$$

Since the oscillators $\alpha_n^I$ and $\alpha_m^J$ only have a non-vanishing commutator if $m = -n$, we find that the normal ordered form of the remaining transverse VIRASORO operators is just

$$L_m^\perp = \frac{1}{2}\sum_{k=0}^{\infty} \alpha_{m-k}^I \alpha_k^I + \frac{1}{2}\sum_{k=1}^{\infty} \alpha_k^I \alpha_{m-k}^I, \qquad m \neq 0.$$

We now want to determine the commutation relations for the transverse VIRASORO modes. First, we will deduce their commutator with the oscillators $\alpha_n^I$

$$\begin{aligned}
[L_m^\perp, \alpha_n^J] &= \frac{1}{2}\sum_{p\in\mathbb{Z}} \left[\alpha_{m-p}^I \alpha_p^I, \alpha_n^J\right] \\
&= \frac{1}{2}\sum_{p\in\mathbb{Z}} \left(\alpha_{m-p}^I \left[\alpha_p^I, \alpha_n^J\right] + \left[\alpha_{m-p}^I, \alpha_n^J\right]\alpha_p^I\right) \\
&= \frac{1}{2}\sum_{p\in\mathbb{Z}} \left(\alpha_{m-p}^J \delta_{p+n,0}\, p + \alpha_p^J \delta_{m-p+n,0}\,(m-p)\right) \\
&= -n\,\alpha_{m+n}^J.
\end{aligned} \tag{35}$$

Since we set $a$ to a real number, the newly defined $L_0^\perp$ also satisfies the relation above with $m = 0$. With eq. (31) we find the commutator

$$\begin{aligned}
[L_m^\perp, x_0^I] &= \frac{1}{2}\sum_{p\in\mathbb{Z}} \left[\alpha_{m-p}^J \alpha_p^J, x_0^I\right] \\
&= \frac{1}{2}\sum_{p\in\mathbb{Z}} \left(\alpha_{m-p}^J \left[\alpha_p^J, x_0^I\right] + \left[\alpha_{m-p}^J, x_0^I\right]\alpha_p^J\right) \\
&= \frac{1}{2}\sum_{p\in\mathbb{Z}} \left(\alpha_{m-p}^J (-\imath)\sqrt{2\alpha'}\,\eta^{IJ}\delta_{p,0} + \alpha_p^J (-\imath)\sqrt{2\alpha'}\,\eta^{IJ}\delta_{m-p,0}\right) \\
&= -\imath\sqrt{2\alpha'}\,\alpha_m^I.
\end{aligned} \tag{36}$$

With eq. (35) we finally deduce the commutator between two transverse VIRASORO operators

$$\begin{aligned}
[L_m^\perp, L_n^\perp] &= \frac{1}{2}\sum_{k\geq 0} [\alpha_{m-k}^I \alpha_k^I, L_n^\perp] + \frac{1}{2}\sum_{k<0} [\alpha_k^I \alpha_{m-k}^I, L_n^\perp] \\
&= \frac{1}{2}\sum_{k\geq 0} \alpha_{m-k}^I [\alpha_k^I, L_n^\perp] + \frac{1}{2}\sum_{k\geq 0} [\alpha_{m-k}^I, L_n^\perp]\,\alpha_k^I + \frac{1}{2}\sum_{k<0} \alpha_k^I [\alpha_{m-k}^I, L_n^\perp] + \frac{1}{2}\sum_{k<0} [\alpha_k^I, L_n^\perp]\,\alpha_{m-k}^I \\
&= \frac{1}{2}\sum_{k\geq 0} \alpha_{m-k}^I\, k\alpha_{n+k}^I + \frac{1}{2}\sum_{k\geq 0} (m-k)\alpha_{m-k+n}^I \alpha_k^I + \frac{1}{2}\sum_{k<0} \alpha_k^I (m-k)\alpha_{m-k+n}^I + \frac{1}{2}\sum_{k<0} k\alpha_{k+n}^I \alpha_{m-k}^I
\end{aligned}$$

To ensure that the oscillators are normal ordered, we distinguish two cases

$\underline{m+n\neq 0}$ In this case, all oscillators in the equation above commute. Thus we can rewrite is as

$$\begin{aligned}[L_m^\perp, L_n^\perp] &= \frac{1}{2}\sum_{k\in\mathbb{Z}}(m-k)\alpha^I_{m+n-k}\alpha^I_k + \frac{1}{2}\sum_{k\in\mathbb{Z}} k\alpha^I_{m-k}\alpha^I_{k+n} \\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}}(m-k)\alpha^I_{m+n-k}\alpha^I_k + \frac{1}{2}\sum_{k\in\mathbb{Z}}(k-n)\alpha^I_{m-k+n}\alpha^I_k \\ &= \frac{m-n}{2}\sum_{k\in\mathbb{Z}}\alpha_{m+n-k}\alpha^I_k \\ &= (m-n)\,L^\perp_{m+n}. \end{aligned} \tag{37}$$

$\underline{m+n=0}$ We write $n=-m$ and receive

$$\begin{aligned}[L_m^\perp, L_{-m}^\perp] &= \frac{1}{2}\sum_{k\geq 0}(m-k)\alpha^I_{-k}\alpha^I_k + \frac{1}{2}\sum_{k<0}(m-k)\alpha^I_k\alpha^I_{-k} + \frac{1}{2}\sum_{k\geq 0}k\alpha^I_{m-k}\alpha^I_{-m+k} + \frac{1}{2}\sum_{k<0}k\alpha^I_{-m+k}\alpha^I_{m-k} \\ &= \frac{1}{2}\sum_{k=0}^{\infty}(m-k)\alpha^I_{-k}\alpha^I_k + \frac{1}{2}\sum_{k=1}^{\infty}(m+k)\alpha^I_{-k}\alpha^I_k + \frac{1}{2}\sum_{k=-m}^{\infty}(m+k)\alpha^I_{-k}\alpha^I_k + \frac{1}{2}\sum_{k=m+1}^{\infty}(m-k)\alpha^I_{-k}\alpha^I_k. \end{aligned}$$

Without loss of generality we assume $m>0$. The other cases follow by inverting the sign of the commutator. Now all terms except the third one are already normal ordered. We expand the problematic term and find

$$\frac{1}{2}\sum_{k=0}^{m}(m-k)\alpha^I_k\alpha^I_{-k} + \frac{1}{2}\sum_{k=1}^{\infty}(m+k)\alpha^I_{-k}\alpha^I_k = \frac{1}{2}\sum_{k=0}^{m}(m-k)\left[\alpha^I_k, \alpha^I_{-k}\right] + \frac{1}{2}\sum_{k=0}^{m}(m-k)\alpha^I_{-k}\alpha^I_k + \frac{1}{2}\sum_{k=1}^{\infty}(m+k)\alpha^I_{-k}\alpha^I_k.$$

Inserting this equation into the commutator yields

$$[L_m^\perp, L_{-m}^\perp] = \sum_{k=0}^{\infty}(m-k)\alpha^I_{-k}\alpha^I_k + \sum_{k=1}^{\infty}(m+k)\alpha^I_{-k}\alpha^I_k + (D-2)\,A(m)$$

with the function $A(m)$ given by

$$\begin{aligned} A(m) &= \frac{1}{2}\sum_{k=0}^{m}k(m-k) = \frac{m}{2}\sum_{k=1}^{k} - \frac{1}{2}\sum_{k=1}^{m}k^2 \\ &= \frac{m^2}{4}(m+1) - \frac{1}{12}(2m^3+3M^2+m) = \frac{1}{12}(m^3-m). \end{aligned}$$

We conclude that the commutator with $m=-n$ is given by

$$\begin{aligned}[L_m^\perp, L_{-m}^\perp] &= 2m\left(\frac{1}{2}\alpha^I_0\alpha^I_0 + \frac{1}{2}\sum_{k=1}^{\infty}\alpha^I_{-k}\alpha^I_k\right) + \frac{D-2}{12}(m^3-m) \\ &= 2m\,L^\perp_0 + \frac{D-2}{12}(m^3-m). \end{aligned}$$

Combining the results of both cases we receive the final form

$$[L_m^\perp, L_n^\perp] = (m-n)\,L^\perp_{m+n} + \frac{D-2}{12}(m^3-m)\,\delta_{m+n,0}. \tag{38}$$

This defines a Lie algebra, which is verified in appendix A.2 and is called the Virasoro algebra.

### 4.4 Lorentz Generators

In section 2.6 we found that the conserved charges induced by the LORENTZ symmetry for open strings are of the form

$$M^{\mu\nu} = \int_0^\pi \mathrm{d}\sigma\, M^{\mu\nu\tau}(\tau,\sigma) = \int_0^\pi \mathrm{d}\sigma\, \left(X^\mu \mathscr{P}^{\tau\nu} - X^\nu \mathscr{P}^{\mu\tau}\right) = \frac{1}{2\pi\,\alpha'} \int_0^\pi \mathrm{d}\sigma\, \left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\right).$$

Since $\{\cos(n\sigma) \,|\, n \in \mathbb{N}_0\}$ form an orthogonal set with respect to the inner product $\langle\phi|\psi\rangle = \int_0^\pi \mathrm{dvol}\phi\cdot\psi$, we find that the only contributing terms are

$$\int_0^\pi \mathrm{d}\sigma\, X^\mu \dot{X}^\nu = \int_0^\pi \mathrm{d}\sigma \left( x_0^\mu \sqrt{2\alpha'}\, \alpha_0^\nu + 2\alpha'\, \alpha_0^\mu \alpha_0^\nu \tau^2 + 2\alpha'\, \imath \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n}\, \alpha_n^\mu\, \alpha_{-n}^\nu \cos^2(n\sigma) \right.$$
$$\left. + 2\alpha'\, \imath \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n}\, \alpha_n^\mu\, \alpha_n^\nu \cos^2(n\sigma)\, e^{-2\imath n\tau} \right),$$

where one should note that the first and the third term are the only non-symmetric terms under an exchange of $\mu, \nu$. It follows that

$$M^{\mu\nu} = \frac{1}{2\pi\,\alpha'} \int_0^\pi \mathrm{d}\sigma \left( x_0^\mu \sqrt{2\alpha'}\alpha_0^\nu - x_0^\nu \sqrt{2\alpha'}\, \alpha_0^\mu + 2\alpha'\, \imath \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n}\, \alpha_n^\mu \alpha_{-n}^\nu \cos^2(n\sigma) - 2\alpha'\, \imath \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n}\, \alpha_n^\nu \alpha_{-n}^\mu \cos^2(n\sigma) \right)$$

and with $\left[\alpha_n^\mu, \alpha_{-n}^\nu\right] \propto \delta_{\mu\nu} = 0$

$$\begin{aligned}
M^{\mu\nu} &= x_0^\mu\, p^\nu - x_0^\nu\, p^\mu + \frac{\imath}{\pi} \int_0^\pi \mathrm{d}\sigma \left( \sum_{n=1}^\infty \frac{1}{n} \left(\alpha_n^\mu\, \alpha_{-n}^\nu - \alpha_n^\nu\, \alpha_{-n}^\mu\right) \cos^2(n\sigma) - \sum_{n=1}^\infty \frac{1}{n} \left(\alpha_{-n}^\mu\, \alpha_n^\nu - \alpha_{-n}^\nu\, \alpha_n^\mu\right) \cos^2(n\sigma) \right) \\
&= x_0^\mu\, p^\nu - x_0^\nu\, p^\mu - \frac{2\imath}{\pi} \int_0^\pi \mathrm{d}\sigma \sum_{n=1}^\infty \frac{1}{n} \left(\alpha_{-n}^\mu\, \alpha_n^\nu - \alpha_{-n}^\nu\, \alpha_n^\mu\right) \cos^2(n\sigma) \\
&= x_0^\mu\, p^\nu - x_0^\nu\, p^\mu - \imath \sum_{n=1}^\infty \frac{1}{n} \left(\alpha_{-n}^\mu\, \alpha_n^\nu - \alpha_{-n}^\nu\, \alpha_n^\mu\right). \qquad (39)
\end{aligned}$$

This form is normal ordered for all indices $\mu, \nu$. The LORENTZ algebra is generated by the $\frac{D(D-1)}{2}$ antisymmetric hermitian elements

$$M^{\mu\nu} = \left(-\imath\delta^{\mu i}\, \delta^{\nu j} + \imath\delta^{\mu j}\delta^{\nu i}\right)_{0\le i,j\le d}, \qquad 0 \le \mu < \nu \le d.$$

These generators satisfy the following commutation relation

$$\left[M^{\mu\nu}, M^{\rho\sigma}\right] = \imath\eta^{\mu\rho}\, M^{\nu\sigma} - \imath\eta^{\nu\rho}\, M^{\mu\sigma} + \imath\eta^{\mu\sigma}\, M^{\rho\nu} - \imath\eta^{\nu\sigma}\, M^{\rho\mu}$$

and especially

$$\left[M^{-I}, M^{-J}\right] = 0.$$

The LORENTZ generators we constructed in eq. (39) must fulfil these conditions, since otherwise our theory would not be LORENTZ invariant. This condition will only be satisfied for a specific number of dimensions $D$, as we will derive in the following. The respective operator is given by

$$M^{-I} = l^{-I} + E^{-I}$$

with

$$\begin{aligned}
l^{\mu\nu} &= x_0^\mu\, p^\nu - x_0^\nu\, p^\mu \\
E^{\mu\nu} &= -\imath \sum_{n=1}^\infty \frac{1}{n} \left(\alpha_{-n}^\mu\, \alpha_n^\nu - \alpha_{-n}^\nu\, \alpha_n^\mu\right)
\end{aligned}$$

Since $x_0^I$ and $p^- = \frac{1}{2p^+}(p^I p^I + m^2)$ do not commute, we first have to ensure that the operator $M^{-I}$ is hermitian by symmetrising the respective term

$$l^{-I} = x_0^- p^I - \frac{1}{2}\left(x_0^I p^- + p^- x_0^I\right).$$

With eq. (33) and eq. (34) we find that the correct form of the operator is

$$M^{-I} = x_0^- p^I - \frac{1}{4\alpha' p^+}\left(x_0^I\left(L_0^\perp + a\right) + \left(L_0^\perp + a\right)x_0^I\right) - \frac{\imath}{\sqrt{2\alpha'}\,p^+}\sum_{n=1}^{\infty}\frac{1}{n}\left(L_{-n}^\perp \alpha_n^I - \alpha_{-n}^I L_n^\perp\right).$$

The commutator consist of four terms

$$\left[M^{-I}, M^{-J}\right] = \left[l^{-I}, l^{-J}\right] + \left[l^{-I}, E^{-J}\right] + \left[E^{-I}, l^{-J}\right] + \left[E^{-I}, E^{-J}\right]$$

which we will examine separately. Since the first term is simply the analogue commutator for a free particle, it vanishes. Therefore, the commutator $\left[M^{-I}, M^{-J}\right]$ can only exist of terms being quadratic or quartic in oscillators. The terms with four oscillators are the ones appearing in the classical theory and thus must vanish as well. The antisymmetry of the Lie bracket and the form of $E^{-I}$ imply that the commutator has to be of the form

$$\left[M^{-I}, M^{-J}\right] = -\frac{1}{2\alpha' p^{+2}}\sum_{n=1}^{\infty}\Delta_n\left(\alpha_{-n}^I \alpha_n^J - \alpha_{-n}^J \alpha_n^I\right),$$

for some constants $\Delta_m$. We will calculate the quantity

$$\begin{aligned}
\Lambda^{IJ} &:= 2\alpha' p^{+2}\langle 0|\alpha_m^I\left[M^{-I}, M^{-J}\right]\alpha_{-m}^J|0\rangle \\
&= -\left\langle 0\middle|\alpha_m^I\sum_{n=1}^{\infty}\Delta_n\left(\alpha_{-n}^I\alpha_n^J - \alpha_{-n}^J\alpha_n^I\right)\alpha_{-m}^J\middle|0\right\rangle \\
&= -\left\langle 0\middle|\sum_{n=1}^{\infty}\Delta_n\alpha_m^I\alpha_{-n}^I\alpha_n^J\alpha_{-m}^J\middle|0\right\rangle \\
&= -\left\langle 0\middle|\sum_{n=1}^{\infty}\Delta_n\alpha_m^I\alpha_{-n}^I\left[\alpha_n^J, \alpha_{-m}^J\right]\middle|0\right\rangle \\
&= -\langle 0|m\,\Delta_m\alpha_m^I\alpha_{-m}^I|0\rangle = -m^2\,\Delta_m.
\end{aligned}$$

Since $\left[l^{-I}, l^{-J}\right]$ is the Lorentz algebra commutator of the quantum point particle it vanishes. We derive the two remaining terms seperately.

### Commutator $\left[E^{-I}, E^{-J}\right]$

For the sake of clearness, we will define three operators in the same way as in [2] on page 23

$$\begin{aligned}
\left|A^J\right\rangle &:= \sum_{l=1}^{\infty}\frac{1}{l}\alpha_{-l}^J L_l^\perp \alpha_{-m}^J|0\rangle = m\sum_{l=1}^{m}\frac{1}{l}\alpha_{-l}^J\alpha_{l-m}^J|0\rangle, \\
\left|B^J\right\rangle &:= \sum_{l=1}^{\infty}\frac{1}{l}L_{-l}^\perp\alpha_l^J\alpha_{-m}^J|0\rangle = L_{-m}^\perp|0\rangle, \\
\left|C^{IJ}\right\rangle &:= \sum_{l=1}^{\infty}\frac{1}{l}\left(\alpha_{-l}^I L_l^\perp - L_{-l}^\perp\alpha_l^I\right)\alpha_{-m}^J|0\rangle = \sum_{l=1}^{\infty}\frac{1}{l}\alpha_{-l}^I L_l^\perp\alpha_{-m}^J|0\rangle = m\sum_{l=1}^{m}\alpha_{-l}^I\alpha_{l-m}^J|0\rangle
\end{aligned}$$

and find

$$\left|A^J\right\rangle - \left|B^J\right\rangle = \sum_{l=1}^{\infty}\frac{1}{l}\left(\alpha_{-l}^J L_l^\perp - L_{-l}^\perp\alpha_l^J\right)\alpha_{-m}^J|0\rangle = -\sqrt{2\alpha'}p^+\,\imath\,E^{-J}\alpha_{-m}^J|0\rangle$$

and furthermore that the first term of the commutator is given by

$$2\alpha' p^{+2}\langle 0|\alpha_m^I E^{-I}E^{-J}\alpha_{-m}^J|0\rangle = \left(\left\langle A^I\right| - \left\langle B^I\right|\right)\left(\left|A^J\right\rangle - \left|B^J\right\rangle\right) = \left\langle A^I\middle|A^J\right\rangle + \left\langle B^I\middle|B^J\right\rangle - \left\langle A^I\middle|B^J\right\rangle - \left\langle B^I\middle|A^J\right\rangle.$$

The second term then is simply given by

$$2\alpha' p^{+2} \langle 0|\alpha^I_m E^{-J} E^{-I} \alpha^J_{-m}|0\rangle = \langle C^{JI}|C^{IJ}\rangle.$$

We calculate

$$\begin{aligned}
\langle A^I|A^J\rangle &= m^2 \left\langle 0\left|\sum_{l,n=1}^{m} \frac{1}{ln}\alpha^I_{m-l}\alpha^I_l\alpha^J_{-n}\alpha^J_{n-m}\right|0\right\rangle = 0 \\
\langle B^I|B^J\rangle &= \langle 0|L^\perp_m L^\perp_{-m}|0\rangle = \langle 0|[L^\perp_m, L^\perp_{-m}]|0\rangle = \frac{D-2}{12}(m^3-m) \\
\langle C^{JI}|C^{IJ}\rangle &= m^2 \left\langle 0\left|\sum_{l,n=1}^{m} \frac{1}{ln}\alpha^I_{m-l}\alpha^J_l\alpha^I_{-n}\alpha^J_{n-m}\right|0\right\rangle \\
&= m^2 \left\langle 0\left|\sum_{l,n=1}^{m} \frac{1}{ln}\left[\alpha^I_{m-l}, \alpha^I_{-n}\right]\alpha^J_l\alpha^J_{n-m}\right|0\right\rangle \\
&= m^2 \left\langle 0\left|\sum_{l=1}^{m-1} \frac{1}{l(m-l)}(m-l)\alpha^J_l\alpha^J_{-l}\right|0\right\rangle \\
&= m^2 \sum_{l=1}^{m-1} 1 = m^2(m-1)
\end{aligned}$$

and

$$\begin{aligned}
\langle A^I|B^J\rangle &= m\left\langle 0\left|\sum_{l=1}^{m}\alpha^I_{m-l}\alpha^I_l\frac{1}{l}L^\perp_{-m}\right|0\right\rangle \\
&= m\left\langle 0\left|\sum_{l=1}^{m}\frac{1}{l}\alpha^I_{m-l}\left[\alpha^I_l, L^\perp_{-m}\right]\right|0\right\rangle \\
&= m\left\langle 0\left|\sum_{l=1}^{m}\left[\alpha^I_{m-l}, \alpha^I_{l-m}\right]\right|0\right\rangle \\
&= m^2\sum_{l=1}^{m}(m-l) = \frac{m^2}{2}(m-1) = \langle B^I|A^J\rangle.
\end{aligned}$$

Collecting all the terms yields the result

$$2\alpha' p^{+2}\langle 0|\alpha^I_m\left[E^{-I}, E^{-J}\right]\alpha^J_{-m}|0\rangle = \frac{D-2}{12}(m^3-m) - 2m^2(m-1). \tag{40}$$

### Commutator $\left[l^{-I}, E^{-J}\right]$

To derive the second commutator, we introduce a fourth state

$$\left|D^{IJ}\right\rangle := \imath\sqrt{2\alpha'}\, p^+ l^{-I}\alpha^J_{-m}|0\rangle.$$

As we have seen before, we have to symmetrise the angular momentum $l^{-I}$ to ensure that it is hermitian. We rewrite the operator as

$$\begin{aligned}
l^{-I} &= x^-_0 p^I - \frac{1}{2}\left(x^I_0 p^- + p^- x^I_0\right) \\
&= x^-_0 p^I - \frac{1}{4\alpha' p^+}\left(x^I_0\left(L^\perp_0 + a\right) + \left(L^\perp_0 + a\right)x^I_0\right) \\
&= x^-_0 p^I - \frac{1}{4\alpha' p^+}\left(2x^I_0\left(L^\perp_0 + a\right) + \left[L^\perp_0 + a, x^I_0\right]\right) \\
&= \left(x^-_0 + \frac{\imath}{2p^+}\right)p^I - \frac{1}{2\alpha' p^+}x^I_0\left(L^\perp_0 + a\right).
\end{aligned}$$

Since the momentum operator $p^I$ commutes with all oscillators and annihilates the vacuum state, we find for $|D^{IJ}\rangle$

$$
\begin{aligned}
|D^{IJ}\rangle &= -\frac{\imath}{\sqrt{2\alpha'}} x_0^I \left(L_0^\perp + a\right) \alpha_{-m}^J |0\rangle \\
&= -\frac{\imath}{\sqrt{2\alpha'}} (m+a) x_0^I \alpha_{-m}^J |0\rangle .
\end{aligned}
$$

The quantity under consideration now is given by

$$
2\alpha' p^{+2} \langle 0|\alpha_m^I \left[l^{-I}, E^{-J}\right] \alpha_{-m}^J |0\rangle = \langle D^{II}| \left(|B^J\rangle - |A^J\rangle\right) + \langle C^{JI}|D^{IJ}\rangle .
$$

Whenever there is an odd number of oscillators in an operator, the expectation value with respect to the vacuum state vanishes. Thus, the only non-zero term in the sum above is $\langle D^{II}|B^J\rangle$ with

$$
\begin{aligned}
\langle D^{II}|B^J\rangle &= \frac{\imath}{\sqrt{2\alpha'}} (m+a) \langle 0|\alpha_m^I x_0^I L_{-m}^\perp|0\rangle \\
&= -(m+a) \langle 0|\alpha_m^I \alpha_{-m}^I|0\rangle = -m(m+a).
\end{aligned} \tag{41}
$$

The opposite commutator $\left[E^{-I}, l^{-J}\right]$ is given by the expression

$$
2\alpha' p^{+2} \langle 0|\alpha_m^I \left[E^{-I}, l^{-J}\right]|0\rangle = \left(\langle B^I| - \langle A^I|\right) |D^{JJ}\rangle + \langle D^{JI}|C^{IJ}\rangle = -m(m+a). \tag{42}
$$

We insert eq. (40), eq. (41) and eq. (42) into the expression of $\Lambda^{IJ}$ and deduce that the coefficients $\Delta_m$ read

$$
\Delta_m = -\frac{1}{m^2}\Lambda^{IJ} = m\left(\frac{D-2}{12} - 2\right) - \frac{1}{m}\left(\frac{D-2}{12} + 2a\right)
$$

As we have mentioned before, for bosonic string theory to be Lorentz invariant, the commutator $\left[M^{-I}, M^{-J}\right]$ must vanish. We conclude that the dimension of spacetime is $D = 26$ and in accordance with eq. (34) we find $a = -1$.

## References


[1] B. Zwiebach,
*A First Course in String Theory*, ed.2,
2009.

[2] C. A. Keller,
*Introduction to String Theory*,
2016.

# A Appendix

## A.1 Analytical Continuation of the $\zeta$-Function

We will first derive the analytical continuation of the gamma function

$$\Gamma(z) = \int_0^\infty \mathrm{d}t\, e^{-t}\, t^{z-1}, \qquad \mathrm{Re}(z) > 0.$$

We find that

$$\Gamma(z) = \int_0^1 \mathrm{d}t\, t^{z-1}\left(e^{-t} - \sum_{n=0}^{N} \frac{(-t)^n}{n!}\right) + \sum_{n=0}^{N} \frac{(-1)^n}{n!}\frac{1}{z+n} + \int_1^\infty \mathrm{d}t\, e^{-t}\, t^{z-1},$$

since

$$\int_0^1 \mathrm{d}t\, t^{z-1} \sum_{n=0}^{N} \frac{(-t)^n}{n!} = \sum_{n=0}^{N} \frac{(-1)^n}{n!} \int_0^1 \mathrm{d}t\, t^{z+n-1} = \sum_{n=0}^{N} \frac{(-1)^n}{n!}\frac{1}{z+n}.$$

The first integral is of the order $\int_0^1 \mathrm{d}t\, t^{z-1}\, t^{N+1}$. For this to converge, we require $\mathrm{Re}(z+N) > -1$, which is equivalent to $\mathrm{Re}(z) > -N-1$. Although having poles at $z = -n,\ n \in \mathbb{N}_0$ with residues

$$\mathrm{Res}_{z=-n}(\Gamma(z)) = \frac{(-1)^n}{n!}$$

the function is well defined and holomorphic on the rest of the complex plane and therefore meromorphic. Next we examine the product

$$\begin{aligned}
\Gamma(s)\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \int_0^\infty \mathrm{d}t\, e^{-t}\, t^{s-1}, \qquad \mathrm{Re}(s) > 1 \\
&\overset{t' := \frac{y}{n}}{=} \sum_{n=1}^{\infty} \frac{1}{n^s}\, n \int_0^\infty \mathrm{d}t'\, e^{-nt'}\, n^{s-1}\, t'^{s-1} \\
&= \sum_{n=1}^{\infty} \int_0^\infty \mathrm{d}t'\, e^{-nt'}\, t'^{s-1} \\
&= \int_0^\infty \mathrm{d}t'\, t'^{s-1} \sum_{n=1}^{\infty} \left(e^{-t'}\right)^n \\
&= \int_0^\infty \mathrm{d}t'\, \frac{t'^{s-1}}{e^{t'}-1}.
\end{aligned}$$

One finds that the Laurent series of $\left(e^{t'}-1\right)^{-1}$ is given by

$$\frac{1}{e^{t'}-1} = \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + \mathcal{O}(t'^2).$$

We can rewrite the product as

$$\Gamma(s)\zeta(s) = \int_0^1 \mathrm{d}t\, t^{s-1}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\right) + \frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)} + \int_1^\infty \mathrm{d}t\, \frac{t^{s-1}}{e^t-1}$$

as one readily proves by integrating the last three terms in the first integral. This yields

$$-\frac{1}{12} = -\mathrm{Res}_{s=-1}(\Gamma(s)\zeta(s)) = -\lim_{s\to-1}\zeta(s)\,\mathrm{Res}_{s=-1}(\Gamma(s)) = \lim_{s\to-1}\zeta(s).$$

## A.2 The Virasoro Algebra

To verify that eq. (38) defines a Lie algebra, we will first introduce the Witt algebra, characterised by

$$\left[L_m^\perp, L_n^\perp\right] = (m-n)L_{m+n}^\perp.$$

In fact, we show that the Lie bracket is antisymmetric

$$\left[L_n^\perp, L_m^\perp\right] = -(m-n)L_{n+m}^\perp = -\left[L_m^\perp, L_n^\perp\right]$$

and that it satisfies the Jacobi identity

$$\begin{aligned}
&\left[L_n^\perp, \left[L_m^\perp, L_k^\perp\right]\right] + \left[L_m^\perp, \left[L_k^\perp, L_n^\perp\right]\right] + \left[L_k^\perp, \left[L_n^\perp, L_m^\perp\right]\right] \\
&\quad = (m-k)\left[L_n^\perp, L_{m+k}^\perp\right] + (k-n)\left[L_m^\perp, L_{k+n}^\perp\right] + (n-m)\left[L_k^\perp, L_{n+m}^\perp\right] \\
&\quad = L_{n+m+k}^\perp\big((m-k)(n-m-k) + (k-n)(m-k-n) + (n-m)(k-n-m)\big) \\
&\quad = 0.
\end{aligned}$$

The Virasoro algebra is the central extension of the Witt algebra. We verify the antisymmetry of eq. (38) by

$$\left[L_n^\perp, L_m^\perp\right] = -(m-n)L_{m+n}^\perp - \frac{D-2}{12}(m^3-m)\,\delta_{m+n,0} = -\left[L_m^\perp, L_n^\perp\right]$$

and

$$\begin{aligned}
&\left[L_n^\perp, \left[L_m^\perp, L_k^\perp\right]\right] + \left[L_m^\perp, \left[L_k^\perp, L_n^\perp\right]\right] + \left[L_k^\perp, \left[L_n^\perp, L_m^\perp\right]\right] \\
&\quad = (m-k)\left[L_n^\perp, L_{m+k}^\perp\right] + (k-n)\left[L_m^\perp, L_{k+n}^\perp\right] + (n-m)\left[L_k^\perp, L_{n+m}^\perp\right] \\
&\quad = \frac{D}{12}\,\delta_{n+m+k,0}\big((n^3-n)(m-k) + (m^3-m)(k-n) + (k^3-k)(n-m)\big) \\
&\quad = \frac{D}{12}\,\delta_{m+n+k,0}\big(n^3(m-k) + m^3(k-n) + k^3(n-m)\big) \\
&\quad = \frac{D}{12}\,\big(-(m+k)^3(m-k) + m^3(2k+m) + k^3(-2m-k)\big) \\
&\quad = 0,
\end{aligned}$$

where we used the Jacobi identity of the Witt algebra and the fact that the commutator with the identity vanishes for all elements.