Twistor strings as iterated integrals

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Chapter 1

Introduction

Physicists are concerned with the mathematical formulation of the laws of nature that govern the behaviour of the material world we perceive. Hitherto, four fundamental forces have been identified that altogether constitute our current picture of the interaction of elementary particles. Three of these four forces, the electromagnetic, weak, and strong force, are quantised and part of the Standard Model, while the gravitational force as the fourth one has not been united with the other three successfully. Each force (at least those in the Standard Model) is transmitted by so-called gauge bosons. Similar to the photon in electromagnetic interactions, the gluon is responsible for the strong force that, for example, binds quarks into neutrons and protons. Due to the non-abelian gauge symmetry, quantum chromodynamics, the theory of the strong force and gluons, is one example of a Yang–Mills-theory.

The main goal of particle physics is the identification of data from collider experiments with a mathematical model governing the process. Clearly, several models may be in accordance with the data of experiments conducted in a certain energy regime. Restraining to the "correct" model requires experiments at energies where the predictions of the models differ. One main objective of theoretical physicists is finding a theory that combines the four fundamental forces. The currently most prominent candidate for a unified description of the fundamental laws including quantum chromodynamics is string theory, a model where each elementary particle is represented by a string, whose vibrational mode determines its physical properties. The target space of this theory consists of ten dimensions, where six of them need to be compactified to restore the perceptible four-dimensional spacetime. Unfortunately, a verification of string theory would require energies the current colliders are not yet able to reach.

Beside the physical motivations for supersymmetry, it is foremost a mathematical construction that simplifies the calculations of particle scatterings. In its simplest formulation, meaning that the theory is invariant under $N = 1$ supersymmetry (Weyl or Majorana) spinor, it states that each boson (fermion) is accompanied by its fermionic (bosonic) superpartner. One eminent toy theory, in the sense that it allows for mathematical investigations but does not describe the physical world, is given by $N = 4$ Super-Yang–Mills-theory, where the only free parameters stem from the gauge group. Occasionally, only the introduction of unnatural symmetries permits a mathematical analysis of a theory. Successful methods may potentially be of use for calculations in the original physical theory.

The research of Feynman and Stückelberg gave rise to a perturbative formalism for calculating particle scatterings. Even though the formalism is predictive, in the sense that it is accurate to an arbitrary precision, the amount of terms that needs to be evaluated quickly exceeds human capabilities and makes computers indispensable. It is therefore desirable to find alternative prescriptions that circumvent these perturbative calculations. Oftentimes, recursive methods can be used to simplify the computation of particle scatterings. For example, the Selberg integrals that will be introduced in chapter 4 exhibit boundary values that correspond to Selberg integrals of different order. Connecting these boundary values via the Drinfeld associator allows for a recursive formalism that makes the evaluation of the Selberg integrals more efficient.

This work is concerned with a possible link between these Selberg integrals and gluon scattering in $N = 4$ Super-Yang–Mills-theory. Chapter 2 contains a brief review over scattering amplitudes in general, maximally-helicity-violating gluon amplitudes, and the Britto–Cachazo–Feng–Witten recursion relations. The physical model we will investigate is the twistor string theory proposed by Berkovits and is reviewed in chapter 3. Chapter 4 summarises the recursive properties of the aforementioned Selberg integral and Drinfeld associator. Finally, in chapter 5, we investigate the algebraic similarities of the gluon amplitudes in twistor string theory and the Selberg integrals.

Chapter 2

Gluon scattering amplitudes

2.1 General formalism

To investigate the behaviour of elementary particles, it is unavoidable to study the interactions between them. The quantity measured in collider experiments is usually the *cross section* of a scattering process. It is the generalisation of a particle's size in a classical theory for point particles underlying interactive forces. Thus, it is closely related to the likelihood for a scattering to happen and yields predictions, in form of the *differential cross section*, on the properties of the final states. The incoming and outgoing particles in a scattering event are considered *external*, which, in contrast to virtual particles or intermediate states, must be on-shell. Choosing the mostly positive metric $\eta = (-1,1,1,1)$, this means that they fulfil the energy-momentum relation

$$p^\mu p_\mu = -m^2,$$

where m denotes the physical mass of the particle.

We only give a brief review of the *S matrix*. An in-depth introduction to the topic can be found in [PS95]. To obtain a meaningful understanding of a scattering process, we must consider the interaction of *asymptotic states* defined far away from the point of interaction. These incoming and outgoing beams $|k_1,\dots,k_n\rangle$ and $\langle p_1,\dots p_m|$ are then translated to a common point in time, which we denote by $|k_1,\dots,k_n\rangle_{\text{in}}$ and ${}_{\text{out}}\langle p_1,\dots p_m|$, respectively. With the time-evolution operator $U(t_0,t) = e^{-\mathrm{i}H(t-t_0)}$, this is to say

$$\begin{aligned}
{}_{\text{out}}\langle p_1,\dots,p_m|k_1,\dots,k_n\rangle_{\text{in}} &= \lim_{T\to\infty} \langle p_1,\dots,p_m|e^{-\mathrm{i}H(2T)}|k_1,\dots,l_n\rangle \\
&\equiv \langle p_1,\dots,p_m|S|k_1,\dots k_n\rangle,
\end{aligned}$$

where we introduced the S matrix in the second line. We want to separate the trivial scattering, i.e. the case of identical in and out states, from the meaningful part in S.

Since the former amounts to the identity in the corresponding Hilbert space, we define the latter as the *T matrix*

$$S = \mathbb{1} - \mathrm{i}T.$$

The *scattering amplitude*, or invariant matrix element, $\mathcal{M}$ of an interaction is given by the expectation value of T with the momentum-conserving δ-function split off

$$\langle p_1, \ldots, p_m | T | k_1, \ldots, k_n \rangle = \delta^{(4)} \left(\sum_i k_i - \sum_j p_j \right) \mathcal{M} \left(\{k_i\}_i \to \{p_j\}_j \right).$$

It completely describes the Hamiltonian-dependent part of the scattering.

We continue the introduction of scattering amplitudes with a short summary of the Feynman prescription. A detailed derivation of the latter is given in [Sch14]. The scattering amplitudes introduced before are closely related to the vacuum expectation value of the fields, whose creation operators generate the external particles. Denoting the ground state by $|\Omega\rangle$ and the time-ordering operator by T, the Lehmann–Symanzik–Zimmermann reduction formula reads [Sch14]

$$\begin{aligned} \langle p_1, \ldots, p_m | S | k_1, \ldots, k_n \rangle = & \left[\prod_{i=1}^{n} \mathrm{i} \int \mathrm{d}x_i^4 e^{-\mathrm{i}k_i \cdot x_i} \left(-\partial_i^2 + m_i^2 \right) \right] \left[\prod_{j=1}^{m} \mathrm{i} \int \mathrm{d}y^4 e^{\mathrm{i}p_j y_j} \left(-\partial_j^2 + m_j^2 \right) \right] \\ & \times \langle \Omega | T \left\{ \phi(x_1) \ldots \phi(x_n) \phi(y_1) \ldots \phi(y_m) \right\} | \Omega \rangle . \end{aligned}$$

The factors $-\partial^2 + m^2$ correspond to $p^2 + m^2$ in momentum space and vanish for on-shell particles. Thus, the only surviving terms in the time-ordered vacuum expectation value are those with poles of the form $(p^2 + m^2)^{-1}$. This is to say that the external particles must be on-shell for the l.h.s. to remain non-trivial. We now shifted the task of computing the entries of the S matrix to evaluating the time-ordered vacuum expectation value from above. In the previous discussion, the fields $\phi(x)$ have all been in the Heisenberg picture. To simplify the calculations, it is necessary to transform them into the interaction picture $\phi_0(x)$, meaning that their evolution in time depends on the free Hamiltonian only. After this transformation, we can write

$$\begin{aligned} & \langle \Omega | T \left\{ \phi(x_1) \ldots \phi(x_n) \phi(y_1) \ldots \phi(y_m) \right\} | \Omega \rangle \\ & \qquad = \frac{\langle 0 | T \left\{ \phi_0(x_1) \ldots \phi_0(x_n) \phi_0(y_1) \ldots \phi_0(y_m) e^{\mathrm{i} \int \mathrm{d}x^4 \mathcal{L}_{\mathrm{int}}[\phi_0]} \right\} | 0 \rangle}{\langle 0 | T \left\{ e^{\mathrm{i} \int \mathrm{d}x^4 \mathcal{L}_{\mathrm{int}}[\phi_0]} \right\} | 0 \rangle}, \end{aligned}$$

where we introduced the free vacuum $|0\rangle$ and the interacting part of the Lagrange density $\mathcal{L}_{\mathrm{int}}$. The expansion of the exponential factor gives rise to infinitely many terms that are calculated using Wick's theorem. The resulting contractions can then be visualised in Feynman diagrams.

This work deals with $N = 4$ Super-Yang–Mills theory (SYM), i.e. a supersymmetric theory with a gauge symmetry group that is in general non-abelian. We will review both Yang–Mills theory and supersymmetry separately. A thorough and more complete introduction can be found in [PS95].

Analogous to the $U(1)$ gauge symmetry in quantum electrodynamics (QED), one might think of a theory with fields that are symmetric under the transformation

$$\psi(x) \to V(x)\psi(x),$$

where $V(x)$ is a unitary irreducible $n \times n$ representation of the gauge group, which we will assume to be a Lie group. This is to say that our symmetry at hand is continuous. Consequently, we may expand the transformation in terms of the Lie algebra generators t^i

$$V(x) = \exp\left(\mathrm{i}\alpha^i(x)t^i\right).$$

In contrast to the single fermionic field in QED, here, ψ represents an n-multiplet $(\psi_1(x),\dots,\psi_n(x))$ of fields. Clearly, a mass term $m^2\overline{\psi}\psi$, where [1] [2] $\overline{\psi} = \psi^\dagger C$, is invariant under the transformation. Due to the *locality* of the transformation, meaning that V may depend on x, the directional derivative of the fields

$$n^\mu \partial_\mu \psi(x) = \lim_{\epsilon\to 0}\frac{\psi(x+\epsilon n) - \psi(x)}{\epsilon}$$

does not satisfy the same transformation law as the fields themselves. To allow for a coupling between the fields and their derivatives, we need to introduce a *comparator* U that cancels the anomalous transformation behaviour of $\psi(x+n\epsilon)$

$$U(y,x) \to e^{\mathrm{i}\alpha^i(y)t^i}U(y,x)e^{-\mathrm{i}\alpha^j(x)t^j}.$$

The partial derivative is replaced by the so-called *covariant* derivative

$$n^\mu D_\mu \psi(x) = \lim_{\epsilon\to 0}\frac{\psi(x+n\epsilon) - U(x+n\epsilon,x)\psi(x)}{\epsilon}, \tag{2.1}$$

which transforms as $D_\mu\psi(x) \to V(x)D_\mu\psi(x)$, rendering an expression like $\mathrm{i}\overline{\psi}D_\mu\gamma^\mu\psi$, with γ^μ denoting the Dirac matrices, gauge invariant. We may expand the comparator $U(x+n\epsilon,x)$ in the parameter ϵ, which gives rise to the *connection* or gauge field $A^i_\mu(x)$

$$U(x+n\epsilon,x) = \mathbb{1} + \mathrm{i}\epsilon n^\mu A^i_\mu(x)t^i + \mathcal{O}(\epsilon^2).$$

[1] The expression $\psi^\dagger\psi$ for a spinor ψ is not Lorentz invariant. To form a Lorentz scalar, $\psi^\dagger$ is acted upon by $C = \mathrm{i}\gamma^0$.

[2] We note that the transformation $V(x)$ rotates the spinor multiplet but not the individual spinors. Therefore, the multiplication with C does not intervene with the symmetry transformation.

Inserting this equality into eq. (2.1) yields for the covariant derivative

$$D_\mu = \partial_\mu - \mathrm{i} A^i_\mu t^i .$$

Similarly to QED, we define the field strength – or rather its Lie algebra decomposition – as

$$F^i_{\mu\nu} t^i = \mathrm{i} \left[D_\mu, D_\nu \right] .$$

Using the *structure* constants f^{abc} defined by $\left[t^a, t^b \right] = \mathrm{i} f^{abc} t^c$, we can express the Lie algebra coefficients of the field strength as

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + f^{abc} A^a_\mu A^b_\nu .$$

We merely note that the field strength itself is not invariant under the gauge transformations

$$F^i_{\mu\nu} t^i \to V(x) F^i_{\mu\nu} t^i V(x)^\dagger .$$

To receive a gauge invariant kinetic term for the gauge fields $A^i_\mu(x)$, we must take the trace over the Lie algebra index i

$$\mathscr{L} = -\frac{1}{2} \mathrm{tr} \left[\left(F^i_{\mu\nu} t^i \right)^2 \right] = -\frac{1}{4} \left(F^i_{\mu\nu} \right)^2 ,$$

where we normalised the generators t^i such that $\mathrm{tr}(t^i t^j) = \frac{1}{2} \delta^{ij}$. The excitations of the gauge fields A^i_μ are called *gluons*, which, in the case of $SU(3)$ as the gauge group, are the exchange bosons for the strong force in quantum chromodynamics.

Besides symmetries whose conserved quantities transform as vectors, such as for example angular momentum, there can be conserved quantities that behave as spinors under the Lorentz group. Naturally, the symmetry will have fermionic character and transforms bosons into fermions and vice versa. The first attempt of such a *supersymmetry* connecting integer and half-integer spin particles, in this case mesons and baryons, was made by Miyazawa [Miy66]. In the 1970s, this symmetry was extended to general bosons and fermions. Following the work of Polchinski in [Pol07], we review (extended) supersymmetry and its multiplets.

We already mentioned that the so-called *supercharges* must transform as a spinor. In $d = 4$ dimensions, this amounts to exactly four (the number of degrees of freedom of a spinor in four dimensions) supercharges. Often, one is using an extended symmetry with N such spinors, yielding $4N$ supercharges in four dimensions. We begin by describing the multiplets of a theory with a single supersymmetry present and perform the generalisation to $N > 1$ afterwards. The $N = 1$ supersymmetry algebra is given by

$$\left\{ Q_\alpha, Q^\dagger_{\dot\beta} \right\} = -2 P_\mu \Gamma^\mu_{\alpha\dot\beta} ,$$
$$\left[P^\mu, Q_\alpha \right] = 0,$$

while $\{Q_\alpha, Q_\beta\} = \left\{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\right\} = 0$. It can be shown [Pol07] that Q and $Q^\dagger$ decrease and increase the spin by one half, respectively. With $Q^2 = Q^{\dagger 2} = 0$, it follows that the multiplet consists of two states whose spin numbers differ by one half. Due to CPT invariance, the multiplet is supplemented by its conjugate multiplet with opposite spin. In a theory with extended supersymmetry, the algebra above is generalised to

$$\begin{aligned} \left\{Q^A_\alpha, Q^{\dagger B}_{\dot\beta}\right\} &= -2P_\mu \delta^{AB}\Gamma^\mu_{\alpha\dot\beta}, \\ \left[P^\mu, Q^A_\alpha\right] &= 0, \end{aligned} \tag{2.2}$$

with indices $A, B \in \{1, \dots, N\}$ and $\left\{Q^A_\alpha, Q^B_\beta\right\} = \left\{Q^{\dagger A}_{\alpha}, Q^{\dagger B}_{\dot\beta}\right\} = 0$ as above. We end up with N raising and N lowering operators that alter the spin by one half. The resulting multiplets consist of 2^N states that are binomially distributed. Omitting helicities larger than two, the two $N = 4$ multiplets are given by

vector multiplet	$(-1, -\frac{1}{2}^4, 0^6, \frac{1}{2}^4, 1)$
supergravity multiplet	$(-2, -\frac{3}{2}^4, -1^6, -\frac{1}{2}^4, 0)+$ c.c.

In the vector multiplet, the two gauge bosons (gluons) with spin ± 1 are complemented by 2×4 *gluinos* with spin $\pm\frac{1}{2}$ and six scalars. This is the field content to be represented in the super-amplitude below.

2.2 Spinor helicity formalism

The Feynman prescription is a powerful and well-understood tool for calculating scattering amplitudes. Due to its accessibility, it usually serves as the first contact point with particle scattering. Nevertheless, for events involving an increasing amount particles, the number of terms to be evaluated quickly prevents the amplitudes to be calculated by hand and numerical methods are oftentimes indispensable. The reason for this complicatedness is, among other things, the gauge choice when establishing the Feynman rules. Only the full amplitude satisfies gauge invariance – single Feynman graphs depend on the gauge choice. In this section, we introduce the *spinor-helicity-formalism*, which expresses the results of the Feynman prescription for scattering amplitudes in gauge theory in a gauge invariant and accessible manner.

We use the convention of a mostly-plus metric, i.e. $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which we often accompany by σ^0 for the identity in two dimensions when dealing with the Pauli matrices. We define the four-vector $\sigma^\mu := (1, \sigma^i)$ together with the bispinor $p_{a\dot b}$

of a complex four-vector p_μ (with $p_\mu p^\mu = -m^2$) as

$$p_{a\dot{b}} \coloneqq p_\mu \left(\sigma^\mu\right)_{a\dot{b}} = \begin{pmatrix} -p^0 + p^3 & p^1 - \mathrm{i}p^2 \\ p^1 + \mathrm{i}p^2 & -p^0 - p^3 \end{pmatrix}.$$

In the 2-spinor formalism, indices are raises and lowered with the $\mathrm{SL}(2,\mathbb{C})$ invariant tensors [3]

$$\varepsilon^{ab} = \varepsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\varepsilon_{ab} = -\varepsilon_{\dot{a}\dot{b}}.$$

Since these matrices are skew-symmetric, we need to choose a convention for the ordering when raising resp. lowering indices. We will stick to "going down" with the indices, i.e.

$$\pi_a \equiv \pi^b \varepsilon_{ba}, \qquad \omega^a \equiv \varepsilon^{ab}\omega_b \tag{2.3}$$

and identically for dotted indices. Thus, the bispinor $p^{\dot{a}b}$ is of the form

$$p^{a\dot{b}} = \varepsilon^{ac}\varepsilon^{\dot{b}\dot{d}} p_{c\dot{d}} = \begin{pmatrix} -p^0 - p^3 & -p^1 - \mathrm{i}p^2 \\ -p^1 + \mathrm{i}p^2 & -p^0 + p^3 \end{pmatrix}.$$

We immediately find

$$\det\left(p_{a\dot{b}}\right) = \det\left(p^{\dot{a}b}\right) = -p_\mu p^\mu = m^2.$$

If $m = 0$, we can decompose the matrix $p^{a\dot{b}}$ into two spinors

$$p^{a\dot{b}} = \pi^a \tilde{\pi}^{\dot{b}} \tag{2.4}$$

With the Levi–Civita tensor ε and the convention in eq. (2.3), we can define $\mathrm{SL}(2,\mathbb{C})$-invariant products[3] for negative and positive chirality spinors

$$\begin{aligned} \langle \pi\omega \rangle &\coloneqq \pi^a \omega_a = \pi^a \omega^b \varepsilon_{ba}, \\ [\tilde{\pi}\tilde{\omega}] &\coloneqq \tilde{\pi}^{\dot{a}} \tilde{\omega}_{\dot{a}} = \tilde{\pi}^{\dot{a}} \tilde{\omega}^{\dot{b}} \varepsilon_{\dot{b}\dot{a}}. \end{aligned}$$

Due to the antisymmetry of ε, both products are antisymmetric under the exchange $\pi \leftrightarrow \omega$. With $p^{a\dot{b}} q_{a\dot{b}} = -2p \cdot q$ [4] it is clear that for two null vectors $p^{a\dot{b}} = \pi^a \tilde{\pi}^{\dot{b}}$ and

[3] Note that we have $g\varepsilon g^T = \varepsilon$ for all $g \in \mathrm{SL}(2,\mathbb{C})$.

[4] This follows from

$$\begin{aligned} p^{a\dot{b}} q_{a\dot{b}} =& (-p^0 - p^3)(-q^0 + q^3) + (-p^1 - \mathrm{i}p^2)(q^1 - \mathrm{i}q^2) \\ &+ (-p^1 + \mathrm{i}p^2)(q^1 + \mathrm{i}q^2) + (-p^0 + p^3)(-q^0 - q^3) \\ =& p^0 q^0 - p^0 q^3 + p^3 q^0 - p^3 q^3 - p^1 q^1 + \mathrm{i}p^1 q^2 - \mathrm{i}p^2 q^1 - p^2 q^2 \\ &- p^1 q^1 - \mathrm{i}p^1 q^2 + \mathrm{i}p^2 q^1 - p^2 q^2 + p^0 q^0 + p^0 q^3 - p^3 q^0 - p^3 q^3 \\ =& 2\left(p^0 q^0 - p^1 q^1 - p^2 q^2 - p^3 q^3\right) = -2p^\mu q_\mu \overset{m=0}{=} -\left(p+q\right)^2 \end{aligned}$$

$q^{a\dot{b}} = \omega^a \widetilde{\omega}^{\dot{b}}$ we have the relation [5]

$$\langle pq \rangle [pq] = -2p^\mu q_\mu = -(p+q)^2 . \tag{2.5}$$

The Mandelstam variables are Lorentz-invariant quantities that encode the energy and momentum of the external particles. They are used primarily for four-particle scatterings $1+2 \to 3+4$ and are given by

$$\begin{aligned} s &= -(p_1+p_2)^2, \\ t &= -(p_1-p_3)^2, \\ u &= -(p_1-p_4)^2, \end{aligned}$$

where the momenta p_1, p_2 and p_3, p_4 are considered incoming and outgoing, respectively. Due to energy-momentum conservation, the variables s, t and u are not independent but obey

$$\begin{aligned} s+t+u &= -p_1^2 - p_2^2 - 2p_1 \cdot p_2 - p_1^2 - p_3^2 + 2p_1 \cdot p_3 - p_1^2 - p_4^2 + 2p_1 \cdot p_4 \\ &= \left(\sum_{i=1}^4 m_i^2 \right) - 2p_1^2 - 2p_1 \cdot p_2 + 2p_1 \cdot \underbrace{(p_3+p_4)}_{=p_1+p_2} \\ &= \sum_{i=1}^4 m_i^2 . \end{aligned}$$

For scatterings with more than four external particles, where all momenta are seen as outgoing, they are generalised to

$$\begin{aligned} s_{i_1 \ldots i_n} &:= -\left(p_{i_1} + \ldots + p_{i_n}\right)^2 \\ &= -\sum_{j<k} 2p_{i_j} \cdot p_{i_l} \\ &= \sum_{j<k} s_{jl}, \end{aligned} \tag{2.6}$$

which allows us to write $\langle pq \rangle [pq] = s_{pq}$.

Going back to the introduction of the two spinors π^a and $\widetilde{\pi}^{\dot{b}}$ in eq. (2.4), we notice that this relation only defines the spinors up to the symmetry

$$\pi \to \alpha \pi, \qquad \widetilde{\pi} \to \frac{1}{\alpha} \widetilde{\pi},$$

where $\alpha \in \mathbb{C}^*$. If the momentum p^μ of the external particle is real, then $p_{a\dot{b}}$ must be hermitian. This on the other hand implies that $\widetilde{\pi} = \overline{\pi}$ and that the scaling factor t must lie

[5] We will often denote the spinor products by the corresponding momentum vector, for example for the given vectors p,q we might write $\langle pq \rangle \equiv \langle \pi\omega \rangle$.

in $U(1)$. With $U(1) \cong SO(2)$, we recover the little group scaling of a massless real particle. Nevertheless, we will for now not impose any reality conditions, which otherwise would for example render the three-point gluon amplitude vanishing. It is customary to perform calculations in complexified spacetime and only afterwards restrict the solution to a certain real slice of it. These slices correspond to different signatures of Minkowski space. A thorough treatment of the complexification and slicing of spacetime into Minkowski, euclidean, and split signature can be found in [Ada18].

In sections 5.1 to 5.3, we will require the spinors π to be real. This is always true for a theory in split signature with the metric $\eta = \text{diag}(-1,-1,1,1)$. The reality condition on the bispinors $p_{a\dot{b}}$ that singles out the real split signature spacetime is given by [Ada18]

$$\overline{p_{a\dot{b}}} = \begin{pmatrix} -\overline{p^0}+\overline{p^3} & \overline{p^1}+\mathrm{i}\overline{p^2} \\ \overline{p^1}-\mathrm{i}\overline{p^2} & -\overline{p^0}-\overline{p^3} \end{pmatrix} \overset{!}{=} p_{a\dot{b}} = \begin{pmatrix} -p^0+p^3 & p^1-\mathrm{i}p^2 \\ p^1+\mathrm{i}p^2 & -p^0-p^3 \end{pmatrix},$$

implying $p^0, p^1, p^3 \in \mathbb{R}$ and $p^2 \in \mathrm{i}\mathbb{R}$. Rotating the second coordinate $q^2 := \mathrm{i}p^2$, allows us to write

$$p_{a\dot{b}}^{\mathbb{R}^{2,2}} = \begin{pmatrix} -p^0+p^3 & p^1-q^2 \\ p^1+q^2 & -p^0-p^3 \end{pmatrix}, \qquad p^0, p^1, q^2, p^3 \in \mathbb{R}. \tag{2.7}$$

We can verify that this corresponds to a split signature spacetime by calculating

$$-p^\mu p_\mu = -\det\left(p_{a\dot{b}}^{\mathbb{R}^{2,2}}\right) = -\left(p^0\right)^2 - \left(p^1\right)^2 + \left(q^2\right)^2 + \left(p^3\right)^2.$$

In the decomposition into spinors $p_{a\dot{b}} = \pi_a \tilde{\pi}_{\dot{b}}$, the reality condition $p_{a\dot{b}} = \overline{p_{a\dot{b}}}$ enforces the spinors π and $\tilde{\pi}$ to be real.

2.3 Twistor space

We give a brief review of the twistor space $\mathbb{PT}$, an open subset of $\mathbb{CP}^3$. References to the topic include [Ada18] and [MW91] for a more extensive introduction.

Elements of the complex projective space $\mathbb{CP}^3$ can be represented as complex vectors $Z^A = (Z^1, Z^2, Z^3, Z^4)$ satisfying

$$Z^A \neq 0, \qquad kZ^A \sim Z^A,\ r \in \mathbb{C}^*.$$

This is to say that two elements are equal if they are complex multiples of each other. We decompose the four coordinates of Z^A into two 2-spinors transforming in opposite helicity representations

$$Z^A = \left(\lambda_a, \mu^{\dot{b}}\right).$$

Each element of the twistor space is assigned to a point $x^{a\dot{b}}$ in (complexified) Minkowski space $\mathbb{M}_{\mathbb{C}}$. This relation is captured in the so-called incidence relation

$$\mu^{\dot{b}} = x^{a\dot{b}}\lambda_a. \tag{2.8}$$

This means that one of the spinors $\mu^{\dot{b}}$ is determined by $x^{a\dot{b}}$ and the second spinor λ_a. Therefore, without the homogeneity of the twistor variables, each point in Minkowski space corresponds to a complex plane $\mathbb{C}^2$. Taking homogeneity into account, each point in Minkowski space corresponds to a Riemann sphere $\mathbb{CP}^1$ in twistor space. The spinors π and $\tilde{\pi}$, decomposing the massless bispinor $p_{a\dot{b}}$ we introduced in the previous section, together form an element of the twistor space.

As we mentioned above, twistor space is only an open subset of $\mathbb{CP}^3$. The specific choice of the subset depends on the signature of the (real) spacetime under consideration. We discussed in the previous subsection how the restriction to split signature forces the spinors to be real. In this case, the twistor space is simply given by the real elements in $\mathbb{PT}_{\mathbb{R}}$.

2.4 Maximally helicity violating amplitudes

It is customary to treat the momenta of all external particles in a scattering process to be outgoing. Consequently, the momentum conserving δ-function reads

$$\delta^{(4)}(p) \equiv \delta^{(4)}\left(\sum_{i=1}^{n} p_i^{\mu}\right) = \delta^{(4)}\left(\sum_{i=1}^{n} \pi_i^a \tilde{\pi}_i^{\dot{a}}\right).$$

One can show that the tree-level scattering amplitudes of gluons with only positive (or only negative) helicities vanish. The Feynman rules for the three and four gluon vertex in Yang–Mills-theory are given by [EH15]

$$V^{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) = \sqrt{2}\left(\eta^{\mu_1\mu_2}p_1^{\mu_3} + \text{cycl. perm.}\right),$$
$$V^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) = \eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4}.$$

These vertices are contracted with the corresponding polarisation vectors of the gluons. The negative/positive polarisation vectors $\epsilon^{\pm}_{a\dot{a}}$ of a particle $p_{a\dot{b}} = \pi_a\tilde{\pi}_{\dot{b}}$ with respect to an arbitrary reference vector $\mu_a\tilde{\mu}_{\dot{a}}$ not aligned with any of the external momenta are given by [Wit04]

$$\epsilon^{-}_{a\dot{a}} = \frac{\pi_a\tilde{\mu}_{\dot{a}}}{[\tilde{\pi}\tilde{\mu}]},$$
$$\epsilon^{+}_{a\dot{a}} = \frac{\mu_a\tilde{\pi}_{\dot{a}}}{\langle\mu\pi\rangle}.$$

Choosing the same reference vector for all particles, we find

$$\epsilon^{-i}_{a\dot{a}}\epsilon^{-j,a\dot{a}} \propto [\widetilde{\mu}\widetilde{\mu}] = 0,$$
$$\epsilon^{+i}_{a\dot{a}}\epsilon^{+j,a\dot{a}} \propto \langle\mu\mu\rangle = 0.$$

Since all terms in the three and four vertex above are proportional to the scalar product of the polarisation vectors, the scattering amplitudes with all particles of the same helicity vanish. Furthermore, if the i-th particle has the opposite helicity, we may choose its momentum to be the reference vector of all other helicities. Then, for $j \neq i$,

$$\epsilon^{\mp i}_{a\dot{a}}\epsilon^{\pm j,a\dot{a}} \propto \begin{cases} [ii] \\ \langle ii\rangle \end{cases} = 0,$$

while $\epsilon^{\pm k}\cdot\epsilon^{\pm l} = 0$ for $k, l \neq 1$. Thus, amplitudes with more than three particles and one with opposite helicity are zero. If we consider the scattering of only three particles where the first one has negative helicity, momentum conservation implies

$$p_3\cdot r_3 = p_3\cdot p_1 = \frac{1}{2}(p_3+p_1)^2 = \frac{1}{2}p_2^2 = 0,$$

which is an inadmissible choice for the third reference vector. We emphasise that in a theory without supersymmetry this only holds at tree-level. As we will show towards the end of this section, in a theory with at least one supersymmetry these statements hold at any loop order. The first non-trivial helicity configuration is the so-called *maximally helicity violating* (MHV) amplitude, which contain exactly two negative (two positive) helicity particles. Since the discussion for cases with positive and negative helicities interchanged is completely analogous, we denote an MHV amplitude as the arrangement $--+\ldots+$, while we write for the scattering of particles $++-\ldots-$ $\overline{\text{MHV}}$. More generally, helicity configurations with k negative gluons will be denoted by N^{k-2}MHV (next-to-MHV).

It was conjectured by Parke and Taylor [PT86] and later shown by Berends and Giele [BG88] that these MHV amplitudes take the simple form

$$A_n^{\text{tree}}\left(1^+,\ldots,i^-,\ldots,j^-,\ldots n^+\right) = \delta^{(4)}(p)\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}, \tag{2.9}$$

which is know as the Parke–Taylor formula.

This work is concerned with the supersymmetric version of Yang–Mills theory, more precisely with $\mathcal{N}=4$ SYM theory, which arises from the introduction of $\mathcal{N}=4$ symmetries between the fermionic and bosonic fields. These symmetries imply immense simplification of the structure of scattering amplitudes. As discussed in section 2.1, the $\mathcal{N}=4$ SYM theory consists of two gluons ($h=\pm1$), eight gluinos ($h=\pm1/2$), and six scalars ($h = 0$). At the end of the section, we will comment on the already mentioned vanishing of the loop amplitudes for helicity arrangements with less than two negative (or positive) gluons. Instead of considering scattering processes of specific particle configurations, the (super-)amplitudes in a supersymmetric theory are most often written as a function of the so-called super-wavefunction incorporating all helicity configurations simultaneously.

With the use of Grassmann variables η^A, $1 \leq A \leq 4$, the super-wavefunction for the $N = 4$ SYM vector multiplet reads

$$\begin{aligned}\Phi(p,\eta) =& G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2!}\eta^A\eta^B S_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C \bar{\Gamma}_{ABC}(p) \\ &+ \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D G^-(p).\end{aligned}$$

Note that the symmetric part of S_{AB} is annihilated by the anticommuting variables $\eta^A\eta^B$. The antisymmetric part of S_{AB} corresponds to the $\frac{4(4-1)}{2} = 6$ scalars.

Instead of considering the scattering of specific particle arrangements, one may consider the amplitude of the full super-wavefunction

$$\mathscr{A}_n(p,\eta) = \mathscr{A}(\Phi_1 \ldots \Phi_n)$$

and select the particles under consideration afterwards. This super-amplitude must transform as a singlet under the global $SU(4)$ R-symmetry [EH15]. It follows that the Grassmann variables η appear in powers of four. We may factor out the supersymmetric MHV tree amplitude by writing [DH09]

$$\mathscr{A}_n = \mathscr{A}_n^{\text{tree}} \mathscr{P}_n,$$

where the factor $\mathscr{P}_n$ is a polynomial in $\eta^4 = \eta^1\eta^2\eta^3\eta^4$

$$\mathscr{P}_n = \mathscr{P}_n^{\text{MHV}} + \mathscr{P}_n^{\text{NMHV}} + \ldots \mathscr{P}_n^{\overline{\text{MHV}}},$$

with $\mathscr{P}_n^{\text{MHV}}$, $\mathscr{P}_n^{\text{NMHV}}$, $\ldots \mathscr{P}_n^{\overline{\text{MHV}}}$ containing $0, 4, \ldots 4n - 16$ Grassmann variables. Clearly, the first term $\mathscr{P}_n^{\text{MHV}}$ is equal to 1. The terms MHV, NMHV, ... we introduced for non-supersymmetric amplitudes above are now defined as amplitudes with 8, 12, ... appearances of the Grassmann variables η_A. For pure gluon scattering, this definition reduces to the non-supersymmetric case. Nevertheless, amplitudes like

$$\begin{aligned}&A_n^{\text{tree}}(G^-, S_{12}, S_{34}, G^+, \ldots, G^+) \\ &A_n^{\text{tree}}(G^-, \Gamma_1, \bar{\Gamma}_2, G^+, \ldots, G^+)\end{aligned}$$

also contain eight ηs and will be called MHV. A general expression for the tree-level amplitude of these super-wavefunctions is given by [DH09]

$$\mathscr{A}_n^{\text{MHV}}(\lambda,\tilde{\lambda},\eta) = \frac{\delta^{(4)}(p)\delta^{(8)}(q)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}, \tag{2.10}$$

where the super-momentum conservation is imposed by the (Grassmann-) δ-function

$$\delta^{(8)}(q) \equiv \delta^{(8)}\left(\sum_{i=1}^{n} q_i^{aA}\right) = \delta^{(8)}\left(\sum_{n=1}^{n} \pi_i^a \eta_i^A\right).$$

Since Grassmann δ-functions let their arguments act multiplicatively on the rest of the amplitude [6] , we may rewrite it as

$$\begin{aligned}\delta^{(8)}(q) &= \prod_{A=1}^{4}\left(\sum_{i=1}^{n}\pi_i^1\eta_i^A\right)\left(\sum_{j=1}^{n}\pi_j^2\eta_j^A\right)\\ &= \prod_{A=1}^{4}\sum_{i,j=1}^{n}\pi_i^1\pi_j^2\eta_i^A\eta_j^A\\ &= \frac{1}{2}\prod_{A=1}^{4}\sum_{i,j=1}^{n}\langle ij\rangle\eta_i^A\eta_j^A.\end{aligned}$$

To extract a certain helicity configuration from the scattering of the super-wavefunctions, we must act on eq. (2.10) with the corresponding Grassmann differential operators

particle	G^+	Γ^A	S^{AB}	$\bar{\Gamma}^{ABC}$	G^-
operator	1	∂_i^A	$\partial_i^A\partial_i^B$	$\partial_i^A\partial_i^B\partial_i^C$	$\partial_i^1\partial_i^2\partial_i^3\partial_i^4$

where the derivatives are taken with respect to the Grassmann variables η_i. For example, the $--+\ldots+$ amplitude is generated by the expression

$$\begin{aligned}A_n^{\text{tree}}(1^+,\ldots,i^-,\ldots,j^-,\ldots n^+) &= \delta^{(4)}(p)\prod_{A=1}^{4}\partial_i^A\partial_j^A\frac{\frac{1}{2}\prod_{B=1}^{4}\sum_{i,j=1}^{n}\langle ij\rangle\eta_i^B\eta_j^B}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}\\ &= \delta^{(4)}(p)\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}\end{aligned}$$

which is in agreement with eq. (2.9).

[6] Using the Berezin integral for anticommuting variables $\int \mathrm{d}\theta = 0$, $\int \mathrm{d}\theta\,\theta = 1$, we find for $a,b\in\mathbb{C}$ and φ a Grassmann variable $\int \mathrm{d}\theta\,(\theta-\varphi)(a\theta+b) = \int \mathrm{d}\theta\,(\theta b + a\theta\varphi) = a\varphi + b$. Since every function $f(\theta)$ is of the form $a\theta+b$, $\theta-\varphi$ may be seen as a δ-function. One should note that the usual identity for $a\in\mathbb{R}^+$ $\delta(ax) = \frac{1}{a}\delta(x)$ is altered to $\delta(a\psi) = a\delta(\psi)$ in the case of anticommuting variables.

Denoting the annihilation operators of the $N=4$ multiplet by $a, a^B, a^{BC}, a^{BCD}, a^{BCDE}$ with antisymmetric indices $B, C, D, E \in \{1,2,3,4\}$, the supercharges defined by the relation eq. (2.2) satisfy the relations [EFK11]

$$\begin{aligned}
&\left[Q^\dagger_A,\, a\right] = 0, && \left[Q^A,\, a\right] = [p\epsilon]\, a^A,\\
&\left[Q^\dagger_A,\, a^B\right] = \langle \epsilon p\rangle \delta^B_A a, && \left[Q^A,\, a^B\right] = [p\epsilon]\, a^{AB},\\
&\left[Q^\dagger_A,\, a^{BC}\right] = 2!\langle \epsilon p\rangle \delta^{[B}_A a^{C]}, && \left[Q^A,\, a^{BC}\right] = [p\epsilon]\, a^{ABC},\\
&\left[Q^\dagger_A,\, a^{BCD}\right] = 3!\langle \epsilon p\rangle \delta^{[B}_A a^{CD]}, && \left[Q^A,\, a^{BCD}\right] = [p\epsilon]\, a^{ABCD},\\
&\left[Q^\dagger_A,\, a^{BCDE}\right] = 4!\langle \epsilon p\rangle \delta^{[B}_A a^{CDE]}, && \left[Q^A,\, a^{BCDE}\right] = 0,
\end{aligned}$$

where $Q^A = -\epsilon^\alpha Q^A_\alpha$, $Q^\dagger_A = \epsilon_{\dot\alpha} Q^{\dagger\dot\alpha}_A$ with the anti-commuting parameters ϵ^α and $\epsilon_{\dot\alpha}$ describing the supersymmetry transformations. Choosing the particles' momenta to be outgoing, the scattering is described by the vacuum expectation values $\langle 0|a^{I_1}(p_1)\dots a^{I_n}(p_n)|0\rangle$, where the multi-indices I_i suit the helicities of the particles. It is customary to assume that the vacuum is supersymmetric, i.e. $Q\,|0\rangle = Q^\dagger\,|0\rangle = 0$. It follows that

$$\begin{aligned}
0 &= \langle 0|\left[Q^{(\dagger)A},\, a^{I_1}(p_1)\dots a^{I_n}(p_n)\right]|0\rangle\\
&= \sum_{i=1}^{n} (-1)^{\sum_{j<i}|a^{I_n}|}\, \langle 0|a^{I_1}_1(p_1)\dots \left[Q^{(\dagger)A},\, a^{I_i}_i(p_i)\right]\dots a^{I_n}_n(p_n)|0\rangle\,.
\end{aligned}$$

Together with the commutation relations above, this implies that the all-plus amplitude vanishes

$$\begin{aligned}
0 &= \langle 0|\left[Q^\dagger_A,\, a^A(p_1)a(p_2)\dots a(p_n)\right]|0\rangle\\
&= \langle \epsilon 1\rangle\, \mathscr{A}\left(G^+(p_1)\dots G^+(p_n)\right).
\end{aligned}$$

Furthermore, we may choose a supersymmetric transformation with $|\epsilon\rangle = |1\rangle$, yielding for $n>3$

$$\begin{aligned}
0 &= \langle 0|\left[Q^\dagger_A,\, a^{BCDE}(p_1)a^A(p_2)a(p_3)\dots a(p_n)\right]|0\rangle\\
&= 4!\underbrace{\langle \epsilon 1\rangle}_{=0}\delta^{[A}_2\, \langle 0|a^{CDE]}(p_1)a^A(p_2)a(p_3)\dots a(p_n)|0\rangle\\
&\quad + \langle \epsilon 2\rangle\, \langle 0|a^{ABCD}(p_1)a(p_2)\dots a(p_n)|0\rangle\\
&= \langle \epsilon 2\rangle\, \mathscr{A}\left(G^-(p_1)G^+(p_2)\dots G^+(p_n)\right).
\end{aligned} \tag{2.11}$$

In the case of $n=3$, 3-particle kinematics is characterised by [EH15]

$$\begin{aligned}
\text{either}\quad & \pi_1 \propto \pi_2 \propto \pi_3,\\
\text{or}\quad & \tilde\pi_1 \propto \tilde\pi_2 \propto \tilde\pi_3
\end{aligned}$$

so that the $-++$ tree amplitude given by

$$A^{\text{tree}}_3\left(G^-(p_1)G^+(p_2)G^+(p_3)\right) = \frac{[23]^3}{[31]\,[12]}$$

satisfies equation eq. (2.11) in both cases. Either $\epsilon \propto \pi_1$ implies that $\epsilon \propto \pi_2$ and thus $\langle \epsilon 2 \rangle = 0$ **or** π_1 is not proportional to π_2 forcing $\tilde{\pi}_1 \propto \tilde{\pi}_2$ and rendering the amplitude vanishing.

2.5 BCFW recursion relations

Following the lines of [Sch14] and [DH09], we review the Britto–Cachazo–Feng–Witten (BCFW) recursion relations derived in [BCF05; Bri+05] and their role in the recursive formulation of the gluon scattering amplitudes.

The derivation of the BCFW recursion relations makes use of the admissibility of complex momenta we mentioned in section 2.2. For a tree-level scattering process with n particles, we perform a shift parameterised by a complex number z of two external particle's momenta according to

$$\hat{\tilde{\pi}}_i = \tilde{\pi}_i + z\tilde{\pi}_j, \qquad \hat{\pi}_j = \pi_j - z\pi_i, \tag{2.12}$$

where the remaining spinors remain invariant. With $\left(p_k^{\alpha\dot{\beta}}\right)^2 = p_k^{\alpha\dot{\beta}} p_{k,\alpha\dot{\beta}}$, the shifted momenta still satisfy the mass-shell condition $\hat{p}_k^2 = 0$. Furthermore, momentum conservation stays unchanged due to

$$\hat{p}_i^{\alpha\dot{\beta}} + \hat{p}_j^{\alpha\dot{\beta}} = p_i^{\alpha\dot{\beta}} + z\pi_i^{\alpha}\tilde{\pi}_j^{\dot{\beta}} + p_j^{\alpha\dot{\beta}} - z\pi_i^{\alpha}\tilde{\pi}_j^{\dot{\beta}} = p_i^{\alpha\dot{\beta}} + p_j^{\alpha\dot{\beta}}.$$

Assuming that the shifted amplitude vanishes as $\lim_{z\to\infty}\mathscr{A}(z) = 0$, we may use the residue theorem to decompose it into

$$0 = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{1}{z}\mathscr{A}(z) = \mathscr{A}(0) + \sum_{z_i}\frac{1}{z_i}\mathrm{Res}_{z=z_i}\left(\mathscr{A}(z)\right), \tag{2.13}$$

where the contour encloses the whole complex plane and the z_i denote the locations of the poles of $\mathscr{A}(z)$. The poles of the shifted amplitude arise from the vanishing of a propagator $\hat{P}(z)$ that lies along the flow of the momentum associated to z, i.e. between the particles i and j. Since our discussion is concerning tree-level diagrams, the momentum $\hat{P}(z)$ connects two separate amplitudes; one of which contains the particle i and one that includes particle j. The splitting of the amplitude is depicted in fig. 2.1. Without loss of generality, we say that $\mathscr{A}_R$ contains the external particles $a,\dots,j,\dots,b$ while the remaining legs $b+1,\dots n,1,\dots i,\dots a-1$ connect to $\mathscr{A}_L$. Assuming that the propagator yielding the pole and connecting the two amplitudes flows from $\mathscr{A}_R$ to $\mathscr{A}_L$, we have

$$\hat{P}(z) = \sum_{k=a}^{b} \pi_k\tilde{\pi}_k - z\pi_i\tilde{\pi}_j.$$

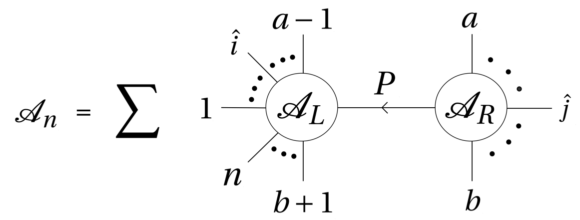

Figure 2.1: BCFW decomposition of the gluon amplitude $\mathscr{A}_n$ in eq. (2.15). The two particles i, j with shifted momenta are part of different subamplitudes $\mathscr{A}_L$ and $\mathscr{A}_R$ connected via P. The sum is taken over all possible decompositions of the amplitude. The shifting of the momenta i and j is chosen such that both subamplitudes are on-shell. It is important to note that the propagator P remains unshifted in the calculation of the amplitude.

For each choice of a and b, the pole condition $\hat{P}^2(z_i) = 0$ (which is equivalent to $\hat{P}^{\alpha\dot{\alpha}}\hat{P}_{\alpha\dot{\alpha}} = 0$) implies for the position of the pole z_i

$$
\begin{aligned}
0 = \frac{1}{2}\hat{P}^{\alpha\dot{\alpha}}(z)\hat{P}_{\alpha\dot{\alpha}}(z) &= \frac{1}{2}\sum_{k,l=a}^{b}\langle kl\rangle\,[kl] - z_i\sum_{k=a}^{b}\langle ik\rangle\,[jk] + \frac{z_i^2}{2}\underbrace{\langle ii\rangle\,[jj]}_{=0} \\
&= -\left(p_a + \ldots + p_b\right)^2 + z_i\sum_{k=a}^{b}\langle ik\rangle\,[kj] \\
\Longrightarrow \qquad z_i &= \frac{\left(p_a + \ldots p_b\right)^2}{\langle ia\rangle\,[aj] + \ldots + \langle ib\rangle\,[bj]},
\end{aligned} \tag{2.14}
$$

where we used eq. (2.5). The terms in the sum of eq. (2.13) thus read

$$
\begin{aligned}
&\frac{1}{z_i}\mathrm{Res}_{z=z_i}\left(\mathscr{A}_L(z)\frac{1}{\left(p_a + \ldots + p_b\right)^2 - z\sum_{k=a}^{b}\langle ik\rangle\,[kj]}\mathscr{A}_R(z)\right) \\
&\qquad = -\mathscr{A}_L(z_i)\frac{1}{\left(p_a + \ldots + p_b\right)^2}\mathscr{A}_R(z_i).
\end{aligned}
$$

Taking into account that the internal particle described by $P(z)$ may acquire different helicities that are to be summed over, eq. (2.13) implies that the physical amplitude $\mathscr{A}(0)$ satisfies the so-called BCFW recursion relation

$$
\begin{aligned}
\mathscr{A}(1,\ldots,n) = &\sum_{a,b\in\{1,\ldots,n\}}\sum_{h}\mathscr{A}_L(\hat{P}^h \to 1,\ldots,a-1,b+1,\ldots,n) \\
&\frac{1}{\left(p_a + \ldots + p_b\right)^2}\mathscr{A}_R(a,\ldots,b \to \hat{P}^{-h}).
\end{aligned} \tag{2.15}
$$

The condition $\lim_{z\to\infty}\mathscr{A}(z) = 0$ mentioned above is crucial for the derivation of the BCFW relations. This restricts the choice of momenta that can be shifted. For example, the shifted tree-level MHV amplitude $(--+\ldots+)$ given in eq. (2.9), where j has negative and i has positive helicity, diverges due to

$$\lim_{z\to\infty} A_n^{\text{tree}}(z) = \lim_{z\to\infty} \frac{\langle\hat{1}2\rangle^4}{\langle\hat{1}2\rangle\ldots\langle i-1,\hat{i}\rangle\langle\hat{i},i+1\rangle\ldots\langle n\hat{1}\rangle} \sim \lim_{z\to\infty} z^2,$$

where we assumed $j=1$. In the case where the shift in i is adjacent to j, i.e. $i \in \{2,n\}$, the divergence is cubic since the factor $\langle\hat{i}\hat{j}\rangle = \langle ij\rangle$ in the denominator does not contribute another power of z. It turns out that the shifts $[ij\rangle = [--\rangle, [-+\rangle$ and $[++\rangle$ yield valid BCFW relations while the case $[+-\rangle$ discussed above is prohibited.

Going back to the super-amplitudes introduced in the previous subsection, the sum over the helicities of the internal particle in eq. (2.15) is replaced by an integral over its four super-variables η_A

$$\mathscr{A} = \sum_{P_i} \int \mathrm{d}^4\eta_{P_i} \mathscr{A}_L(z_i) \frac{1}{P_i^2} \mathscr{A}_R(z_i), \tag{2.16}$$

where, in analogy to eq. (2.15), we denoted $P_i = p_a + \ldots + p_b$. It was shown in [ACK10] that by supplementing the bosonic shift in eq. (2.12) by a suitable transformation of the Grassmann variables, all super-amplitudes vanish in the limit $z \to 0$. In the derivation of the tree-level super amplitudes in [DH09] these transformations read

$$\begin{aligned}
\hat{\tilde{\pi}}_n &= \tilde{\pi}_n + z_i\tilde{\pi}_1, \qquad \hat{\pi}_1 = \pi_1 - z_i\pi_n, \\
\hat{\eta}_n &= \eta_n + z_i\eta_1.
\end{aligned}$$

For example, by using the BCFW relation, the n-particle tree-level NMHV and NN-MHV amplitudes can be expressed as

$$\begin{aligned}
\mathscr{A}_n^{\text{NMHV}} &= \int \frac{\mathrm{d}^4P}{P^2} \int \mathrm{d}^4\eta_P \mathscr{A}_3^{\overline{\text{MHV}}}(z) \mathscr{A}_{n-1}^{\text{NMHV}}(z) \\
&\quad + \sum_{i=4}^{n-1} \int \frac{\mathrm{d}^4P_i}{P_i^2} \int \mathrm{d}^4\eta_{P_i} \mathscr{A}_i^{\text{MHV}}(z_i) \mathscr{A}_{n-i+2}^{\text{MHV}}(z_i),
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
\mathscr{A}_n^{\text{NNMHV}} &= \int \frac{\mathrm{d}^4P}{P^2} \int \mathrm{d}^4\eta_P \mathscr{A}_3^{\overline{\text{MHV}}}(z) \mathscr{A}_{n-1}^{\text{NNMHV}}(z) \\
&\quad + \sum_{i=4}^{n-3} \int \frac{\mathrm{d}^4P_i}{P_i^2} \int \mathrm{d}^4\eta_{P_i} \mathscr{A}_i^{\text{MHV}}(z_i) \mathscr{A}_{n-i+2}^{\text{NMHV}}(z_i) \\
&\quad + \sum_{i=5}^{n-1} \int \frac{\mathrm{d}^4P_i}{P_i^2} \int \mathrm{d}^4\eta_{P_i} \mathscr{A}_i^{\text{NMHV}}(z_i) \mathscr{A}_{n-i+2}^{\text{MHV}}(z_i).
\end{aligned} \tag{2.18}$$

Due to momentum and super-momentum conservation, we may switch to the dual coordinates x and θ defined by

$$\begin{aligned}
x_i - x_{i+1} &= \pi_i\tilde{\pi}_i, \\
\theta_i - \theta_{i+1} &= \pi_i\eta_i,
\end{aligned}$$

which in turn allow us to express the second term of eq. (2.17) as [DH09]

$$\frac{\delta^{(4)}(p)\delta^{(8)}}{\prod_{j=1}^{n}\langle j, j+1\rangle}\sum_{i=4}^{n-1} R_{n;2i}$$

with the superconformal invariants derived in [Dru+10]

$$R_{r;st} = \frac{\langle s, s-1\rangle\langle t, t-1\rangle\,\delta^{(4)}(\Xi_{r;st})}{x_{st}^2\,\langle r|x_{rs}x_{st}|t\rangle\,\langle r|x_{rs}x_{st}|t-1\rangle\,\langle r|x_{rt}x_{ts}|s\rangle\,\langle r|x_{rt}x_{ts}|s-1\rangle},$$

where $x_{ij} \equiv x_i - x_j$. Similar formulae for all tree-level N^pMHV amplitudes were found (cf. [DH09]). The existence of this iterative method for calculating gluon amplitudes motivates the construction of these amplitudes as iterated integrals. A possible relation between the Selberg integrals introduced in the following chapter and the recursive formulation using BCFW relations will be discussed in section 5.4.

Chapter 3

Twistor string theory

In this section we mostly follow the derivation of the tree amplitudes by L. Dolan and P. Goddard [DG07].

3.1 Preliminaries

3.1.1 String action

We consider the world-sheet action for the twistor string introduced by N. Berkovits in [Ber04]

$$S = S_{YZ} + S_{\text{ghosts}} + S_{\text{G}},$$
$$S_{YZ} = \mathrm{i}\int \mathrm{d}^2x\,\sqrt{g}\,\left[Y^{I\mu}D_\mu Z_{IS} + Y^I_\mu \varepsilon^{\mu\nu} D_\nu Z_{IP}\right],$$

where S_G yields a conformal field theory with central charge $c = 28$. The world-sheet scalars Z_S, pseudo-scalars Z_P, and vectors Y_μ are homogeneous coordinates in $\mathbb{CP}^{3|4}$. In comparison to $\mathbb{CP}^3$, $\mathbb{CP}^{3|4}$ is represented by vectors with four bosonic and four fermionic variables, all defined up to a complex scaling factor. Since $D_\mu = \partial_\mu - \mathrm{i}A_\mu$, the reality condition for the field A_μ reads $\overline{A_\mu} \overset{!}{=} -A_\mu$, i.e. it must be imaginary.

The action contains two abelian gauge symmetries. They become apparent when introducing the complex components $z = x_1 + \mathrm{i}x_2$, $Z = Z_S - \mathrm{i}Z_P$, $\widetilde{Z} = Z_S + \mathrm{i}Z_P$, $D_z = \partial_z - \mathrm{i}A_z$, and $A_z = \frac{1}{2}(A_1 - \mathrm{i}A_2)$. The system then is invariant under

$$Y^{\overline{z}} \mapsto g^{-1}Y^{\overline{z}}, \quad Z \mapsto gZ, \quad A_{\overline{z}} \mapsto A_{\overline{z}} - \mathrm{i}g^{-1}\partial_{\overline{z}}g,$$
$$Y^{z} \mapsto \widetilde{g}^{-1}Y^{z}, \quad \widetilde{Z} \mapsto \widetilde{g}\widetilde{Z}, \quad A_{z} \mapsto A_{z} - \mathrm{i}\widetilde{g}^{-1}\partial_{z}\widetilde{g},$$

with $g = e^{\psi+\mathrm{i}\phi}$ and $\widetilde{g} = e^{-\psi+\mathrm{i}\phi}$. Going back to the real coordinates x_i, the transformation for the field A_μ reads

$$A_\mu \mapsto A_\mu + \partial_\mu\phi + \varepsilon_\mu^{\ \nu}\partial_\nu\psi,$$

where the reality condition of the field restricts both ϕ and ψ to be imaginary, thus $\overline{g} = \widetilde{g}$. Consequently, the two symmetries reduce to *one* copy of GL(1, $\mathbb{C}$). This gauge freedom may and will be used to set $A \equiv 0$.

3.1.2 Quantisation and vertices

The mode expansion of the fields Z and Y with conformal weights [Sch08] $\mathscr{I} = 0, 1$ are given by

$$\Phi(z) = \sum_{n \in \mathbb{Z}} \Phi_n z^{-n-\mathscr{I}}, \tag{3.1}$$

where the vacuum satisfies

$$\Phi_n |0\rangle = 0 \qquad \forall n > -\mathscr{I}. \tag{3.2}$$

The canonical commutation relations for the basic fields are given by [1]

$$\begin{aligned} [\![Z^i_m, Y^j_n]\!] &= \delta^{ij} \delta_{m,-n}, \\ \{c_m, b_n\} &= \delta_{m,-n}, \\ \{v_m, u_n\} &= \delta_{m,-n}, \\ [J^A_m, J^B_n] &= \mathrm{i} f^{AB}_C J^C_{m+n} + km\delta_{m,-n}\delta^{AB}. \end{aligned}$$

For the calculation of the vacuum expectation value, we need to introduce the operator e^{q_0} (cf. [DG07] for the full derivation) obeying

$$Z^I_{n+d} e^{dq_0} = e^{dq_0} Z^I_n. \tag{3.3}$$

The reality condition for the field Z^I reads

$$\left(Z^I_n\right)^\dagger = Z^I_{-n}. \tag{3.4}$$

3.2 Vertex operator

The target space of this string theory is $\mathbb{CP}^{3|4}$ with the coordinates

$$Z = \begin{pmatrix} \lambda \\ \mu \\ \psi^I \end{pmatrix},$$

where $\lambda, \mu \in \mathbb{C}^2$ and ψ^I, $I \in \{1,2,3,4\}$ denote homogeneous coordinates, i.e. $kZ \equiv Z$ for $k \in \mathbb{C}^\times$.

[1] The expression $[\![\cdot,\cdot]\!]$ denotes the corresponding Lie-bracket. Here, it denotes the commutator if at least one of i and j is less than 5, and the anticommutator otherwise.

The vertex operator of the gluon state $|\Psi\rangle = f(Z_0)J^A_{-1}|0\rangle$ is

$$V(\Psi, z) = f(Z(z))\, J^A(z),$$

where $J^A(z) = -:Y^A(z)Z^A(z):$ is the normal ordered product of the fields and its vacuum expectation value generates the current algebra correlator. Without accounting for the homogeneity of the coordinates, the wavefunction of the string at $Z' = (\pi, \omega, \theta)$ would be of the form $\Pi_I \delta\left(Z^I(z) - Z'^I\right)$. It turns out that the correct form is [2]

$$W(z) = \int \prod_{a,\dot{a}=1}^{2} \delta\left(k\lambda^a(z) - \pi^a\right) \delta\left(k\mu^{\dot{a}} - \omega^{\dot{a}}\right) \prod_{b=1}^{4} \underbrace{\delta\left(k\psi^b(z) - \theta^b\right)}_{=(k\psi^b(z)-\theta^b)} \frac{\mathrm{d}k}{k}, \tag{3.5}$$

where we used that $\delta\left(k\psi - \theta\right) = k\psi - \theta$ for anticommuting variables ψ and θ. Let us first derive the vertex operator used in Berkovits' article [Ber04]. We use the first δ-function

$$\delta\left(k\lambda^1(z) - \pi^1\right) = \delta\left(\lambda^1(z)\left(k - \frac{\pi^1}{\lambda^1(z)}\right)\right) = \frac{1}{\lambda^1(z)}\delta\left(k - \frac{\pi^1}{\lambda^1(z)}\right)$$

to perform the integration over k, yielding

$$\begin{aligned} W(z) &= \frac{1}{\lambda^1(z)}\frac{\lambda^1(z)}{\pi^1}\delta\left(\frac{\pi^1}{\lambda^1(z)}\lambda^2(z) - \pi^2\right)\prod_{\dot{a}=1}^{2}\delta\left(\frac{\pi^1}{\lambda^1(z)}\mu^{\dot{a}}(z) - \omega^{\dot{a}}\right)\prod_{b=1}^{4}\left(\frac{\pi^1}{\lambda^1(z)}\psi^b(z) - \theta^b\right) \\ &= \frac{1}{\pi^1}\left(\frac{1}{\pi^1}\right)^3 \delta\left(\frac{\lambda^2(z)}{\lambda^1(z)} - \frac{\pi^2}{\pi^1}\right)\prod_{\dot{a}=1}^{2}\delta\left(\frac{\mu^{\dot{a}}(z)}{\lambda^1(z)} - \frac{\omega^{\dot{a}}}{\pi^1}\right)\left(\pi^1\right)^4\prod_{b=1}^{4}\left(\frac{\psi^b(z)}{\lambda^1(z)} - \frac{\theta^b}{\pi^1}\right) \\ &= \delta\left(\frac{\lambda^2(z)}{\lambda^1(z)} - \frac{\pi^2}{\pi^1}\right)\prod_{\dot{a}=1}^{2}\delta\left(\frac{\mu^{\dot{a}}(z)}{\lambda^1(z)} - \frac{\omega^{\dot{a}}}{\pi^1}\right)\prod_{b=1}^{4}\left(\frac{\psi^b(z)}{\lambda^1(z)} - \frac{\theta^b}{\pi^1}\right) \end{aligned}$$

Finally, we multiply by

$$A(\theta) = A_+ + \theta^b A_b + \frac{1}{2}\theta^b\theta^c A_{bc} + \frac{1}{3!}\theta^b\theta^c\theta^d A_{bcd} + \theta^1\theta^2\theta^3\theta^4 A_-,$$

[2] Note that this expression is invariant under $Z(z) \mapsto \alpha(z)Z(z)$ since the substitution $k' = k\alpha(z)$ yields $\frac{\mathrm{d}k}{k} \mapsto \frac{\mathrm{d}k'}{k'}$. Furthermore, it is invariant under $Z' \mapsto \beta Z'$, since it amounts to the substitution $k' = k/\alpha$ after the cancellation of the αs taken out of the δ-functions.

integrate over the Grassmann variables θ^b and Fourier transform in $\omega^{\dot{a}}$ (with parameter $\overline{\pi}_{\dot{a}}$)

$$\begin{aligned}
\widetilde{W}(z) &= \int \mathrm{d}^2\omega\, \mathrm{d}^4\theta\, \delta\left(\frac{\lambda^2(z)}{\lambda^1(z)} - \frac{\pi^2}{\pi^1}\right) \prod_{\dot{a}=1}^{2} \delta\left(\frac{\mu^{\dot{a}}(z)}{\lambda^1(z)} - \frac{\omega^{\dot{a}}}{\pi^1}\right) \prod_{b=1}^{4} \left(\frac{\psi^b(z)}{\lambda^1(z)} - \frac{\theta^b}{\pi^1}\right) \\
&\quad \times A(\theta) \exp\left(\mathrm{i}\omega^{\dot{b}}\overline{\pi}_{\dot{b}}\right) \\
&= \delta\left(\frac{\lambda^2(z)}{\lambda^1(z)} - \frac{\pi^2}{\pi^1}\right) \int \mathrm{d}^2\omega\, \mathrm{d}^4\theta\, \left(\pi^1\right)^2 \prod_{\dot{a}=1}^{2} \delta\left(\frac{\pi^1\mu^{\dot{a}}(z)}{\lambda^1(z)} - \omega^{\dot{a}}\right) \left(\frac{1}{\pi^1}\right)^4 \prod_{b=1}^{4} \left(\frac{\pi^1\psi^b(z)}{\lambda^1(z)} - \theta^b\right) \\
&\quad \times A(\theta) \exp\left(\mathrm{i}\omega^{\dot{b}}\overline{\pi}_{\dot{b}}\right) \\
&= \left(\frac{1}{\pi^1}\right)^2 \delta\left(\frac{\lambda^2(z)}{\lambda^1(z)} - \frac{\pi^2}{\pi^1}\right) \exp\left(\mathrm{i}\frac{\mu^{\dot{a}}(z)\overline{\pi}_{\dot{a}}\pi^1}{\lambda^1(z)}\right) \times \left[A_+ + \frac{\pi^1}{\lambda^1(z)}\psi^b(z)A_b\right. \\
&\quad + \frac{1}{2}\left(\frac{\pi^1}{\lambda^1(z)}\right)^2 \psi^b(z)\psi^c(z)A_{bc} + \frac{1}{3!}\left(\frac{\pi^1}{\lambda^1(z)}\right)^3 \psi^b(z)\psi^c(z)\psi^d(z)A_{bcd} \\
&\quad \left. + \left(\frac{\pi^1}{\lambda^1(z)}\right)^4 \psi^1(z)\psi^2(z)\psi^3(z)\psi^4(z)A_- \right].
\end{aligned}$$

However, for calculating tree amplitudes, we will use the original wavefunction in eq. (3.5), Fourier transform in $\omega^{\dot{a}}$, multiply by $A(\theta)$, and finally integrate over θ^b, yielding

$$\begin{aligned}
\hat{W}(z) &= \int \frac{\mathrm{d}k}{k} \prod_{a=1}^{2} \delta\left(k\lambda^a(z) - \pi^a\right) \exp\left(\mathrm{i}k\mu^{\dot{b}}(z)\overline{\pi}_{\dot{b}}\right) \times \left[A_+ + k\psi^b(z)A_b + \frac{1}{2}k^2\psi^b(z)\psi^c(z)A_{bc}\right. \\
&\quad \left. + \frac{1}{3!}k^3\psi^b(z)\psi^c(z)\psi^d(z)A_{bcd} + k^4\psi^1(z)\psi^2(z)\psi^3(z)\psi^4(z)A_-\right].
\end{aligned}$$

We identify the vertex operators for negative resp. positive helicity gluon states as

$$\begin{aligned}
V_-^A(z) &= \int \mathrm{d}k\, k^3 \prod_{a=1}^{2} \delta\left(k\lambda^a(z) - \pi^a\right) \exp\left(\mathrm{i}k\mu^{\dot{b}}(z)\overline{\pi}_{\dot{b}}\right) J^A(z)\psi^1(z)\psi^2(z)\psi^3(z)\psi^4(z), \\
V_+^A(z) &= \int \frac{\mathrm{d}k}{k} \prod_{a=1}^{2} \delta\left(k\lambda^a(z) - \pi^a\right) \exp\left(\mathrm{i}k\mu^{\dot{b}}(z)\overline{\pi}_{\dot{b}}\right) J^A(z).
\end{aligned}$$

3.3 Tree amplitudes

The tree amplitude of n gluons of degree d (corresponding to $d+1$ negative helicity and $n-d-1$ positive helicity states) is given by

$$\mathscr{A}_n^{\text{tree}} = \int \langle 0|\, e^{dq_0} V_{\epsilon_1}^{A_1} \dots V_{\epsilon_n}^{A_n} |0\rangle \prod_{r=1}^{n} \mathrm{d}z_r \Big/ \mathrm{d}\gamma_M \mathrm{d}\gamma_S, \tag{3.6}$$

where $\mathrm{d}\gamma_M$ and $\mathrm{d}\gamma_S$ are the invariant measures of the Möbius group and the group of scale transformations in twistor superspace respectively. It can be shown (cf. [DG07]) that this expression vanishes unless $|\{i : \epsilon_i = -\}| = d+1$.

3.3.1 Degree zero

Restriction to $n = 3$ Without loss of generality we choose $\epsilon_1 = -$ and $\epsilon_r = +$ for all $r \in \{2,\ldots,n\}$. Due to eq. (3.2) and eq. (3.4), only the zero modes in the expansion of the field Z contribute to the amplitude while the others are annihilated when applied to either the left or right vacuum. The differential $\prod_r \mathrm{d}z_r$ – representing an integration over $Z(z)$ – reduces to an integration over the zero modes of $Z(z)$, i.e. λ_0^a and $\mu_0^{\dot a}$ [3]. **Leaving aside** the current algebra correlator and $\mathrm{d}\gamma_M$, the amplitude is proportional to

$$\int k_1^4 \prod_{r=1}^{n} \frac{\mathrm{d}k_r}{k_r} \prod_{a=1}^{2} \delta\left(\pi_r^a - k_r\lambda^a\right) \prod_{\dot a=1}^{2} \delta\left(\sum_{r=1}^{n} k_r \bar\pi_{r\dot a}\right) \prod_{a=1}^{2} \mathrm{d}\lambda^a \Big/ \mathrm{d}\gamma_S,$$

where the last two δ-functions stem from the integration over μ. This expression includes $n+1$ integrals ($\mathrm{d}^n k\, \mathrm{d}^2\lambda/\mathrm{d}\gamma_S$) and $2n+2$ δ-functions. Thus, after integration we should be left with $n+1$ δ-functions. In four dimensions we expect exactly four δ-functions due to energy-momentum conservation and therefore the amplitude must vanish for $n > 3$.

The amplitude Let us perform the calculation for $n = 3$. We have

$$\begin{aligned}\mathscr{A}^{\text{tree}}_{-++} = \int \prod_{r=1}^{3} \frac{\mathrm{d}k_r}{k_r} \prod_{a=1}^{2} \delta\left(\pi_r^a - k_r\lambda^a\right) \prod_{\dot a=1}^{2} \exp\left(\mathrm{i}k_r \mu^{\dot a}\bar\pi_{r\dot a}\right)\\ \times k_1^4 \underbrace{\langle 0|\, \psi_0^1\psi_0^2\psi_0^3\psi_0^4\, |0\rangle}_{=1}\\ \times \prod_{a,\dot a=1}^{2} \mathrm{d}\lambda^a \mathrm{d}\mu^{\dot a} \frac{f^{A_1A_2A_3}\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3}{(z_1-z_2)(z_2-z_3)(z_3-z_1)} \Big/ \mathrm{d}\gamma_M \mathrm{d}\gamma_S.\end{aligned}$$

The current algebra correlator $\langle 0|\, J_1^A(z_1) J_2^A(z_2) J_3^A(z_3)\, |0\rangle = f^{A_1A_2A_3}(z_1-z_2)^{-1}(z_2-z_3)^{-1}(z_3-z_1)^{-1}$ cancels with the Möbius gauge freedom, allowing us to map any three points on the Riemann sphere to any other three points. Thus,

$$\mathscr{A}^{\text{tree}}_{-++} = \int k_1^4 \prod_{r=1}^{3} \frac{\mathrm{d}k_r}{k_r} \prod_{a=1}^{2} \delta\left(\pi_r^a - k_r\lambda^a\right) \prod_{\dot a=1}^{2} \delta\left(\sum_{r=1}^{n} k_r \bar\pi_{r\dot a}\right) \prod_{a=1}^{2} \mathrm{d}\lambda^a f^{A_1A_2A_3} \Big/ \mathrm{d}\gamma_S,$$

where we performed the integration over $\mu^{\dot a}$. We now make the choice $\mathrm{d}\gamma_S = \mathrm{d}\lambda^1/\lambda^1$ and use the three δ-functions $\delta\left(\pi_r^1 - k_r\lambda^a\right)$ to perform the three integrals over k_r:

$$\begin{aligned}\mathscr{A}^{\text{tree}}_{-++} &= \int k_1^4 \prod_{r=1}^{3} \frac{\mathrm{d}k_r}{k_r} \frac{1}{\lambda^1} \delta\left(\frac{\pi_r^1}{\lambda^1} - k_r\right) \delta\left(\pi_r^2 - \lambda^2 k_r\right) f^{A_1A_2A_3} \prod_{\dot a=1}^{2} \delta\left(\sum_{r=1}^{n} k_r \bar\pi_{r\dot a}\right) \frac{\mathrm{d}\lambda^2\,\mathrm{d}\lambda^1}{\mathrm{d}\lambda^1} \lambda^1\\ &= \left(\frac{\pi_1^1}{\lambda^1}\right)^4 \frac{\lambda_1^3}{\pi_1^1\pi_2^1\pi_2^1} \left(\frac{1}{\lambda^1}\right)^3 f^{A_1A_2A_3} \prod_{\dot a=1}^{2} \delta\left(\sum_{r=1}^{n} \frac{\pi_r^1}{\lambda^1}\bar\pi_{r\dot a}\right) \int \delta\left(\pi_r^2 - \lambda^2\frac{\pi_r^1}{\lambda^1}\right) \mathrm{d}\lambda^2 \lambda^1\\ &= \frac{\left(\pi_1^1\right)^3}{\pi_2^1\pi_3^1} \prod_{\dot a=1}^{2} \delta\left(\sum_{r=1}^{n} \pi_r^1\bar\pi_{r\dot a}\right) \int \delta\left(\pi_r^2 - \lambda^2\frac{\pi_r^1}{\lambda^1}\right) \frac{\mathrm{d}\lambda^2}{\lambda^1}.\end{aligned}$$

[3] In principle we would have to include the integration over the fermionic part ψ^b of Z. Nevertheless, as explained in [DG07] eq.(3.23), the scalar products $\langle 0|\, \psi_0\, |0\rangle$ are equal to one and therefore do not appear in the amplitude.

By using the first three instead of the latter three δ-functions (those with π^2), we ended up with an overall δ-function of the form

$$\delta\left(\sum_{r=1}^{3}\pi_r^1\overline{\pi}_{r\dot{a}}\right) \qquad \text{instead of} \qquad \delta\left(\sum_{r=1}^{3}\pi_r^2\overline{\pi}_{r\dot{a}}\right).$$

These four (counting $a\in\{1,2\}$) δ-functions correspond to momentum conservation. In the following paragraph we want to manipulate the amplitude such that this conservation becomes more apparent.

The factor $\delta\left(\sum_{r=1}^{3}\pi_r^1\overline{\pi}_{r\dot{a}}\right)$ in the amplitude ensures that

$$\sum_{r=1}^{3}\pi_r^2\overline{\pi}_{r\dot{a}} = \sum_{r=1}^{3}\left(\pi_r^2-\pi_r^1\frac{\lambda^2}{\lambda^1}\right)\overline{\pi}_{r\dot{a}}.$$

Now, we can rewrite two of the three δ-functions appearing in the integral over λ^2

$$\prod_{r=2}^{3}\delta\left(\pi_r^2-\pi_r^1\frac{\lambda^2}{\lambda^1}\right)$$

into the "missing" δ-functions

$$\prod_{\dot{a}=1}^{2}\delta\left(\sum_{r=1}^{3}\pi_r^2\overline{\pi}_{r\dot{a}}\right) = \prod_{\dot{a}=1}^{2}\delta\left(\sum_{r=1}^{3}\left(\pi_r^2-\pi_r^1\frac{\lambda^2}{\lambda^1}\right)\overline{\pi}_{r\dot{a}}\right),$$

where we performed the transformation

$$\begin{pmatrix}\pi_2^2-\pi_2^1\frac{\lambda^2}{\lambda^1}\\ \pi_3^2-\pi_3^1\frac{\lambda^2}{\lambda^1}\end{pmatrix} \longrightarrow \begin{pmatrix}\overline{\pi}_{2\dot{1}} & \overline{\pi}_{3\dot{1}}\\ \overline{\pi}_{2\dot{2}} & \overline{\pi}_{3\dot{2}}\end{pmatrix}\begin{pmatrix}\pi_2^2-\pi_2^1\frac{\lambda^2}{\lambda^1}\\ \pi_3^2-\pi_3^1\frac{\lambda^2}{\lambda^1}\end{pmatrix}.$$

Thus, we need to account for the Jacobian determinant of this transformation, i.e. $\overline{\pi}_{2\dot{1}}\overline{\pi}_{3\dot{2}}-\overline{\pi}_{2\dot{2}}\overline{\pi}_{3\dot{1}} = [2,3]$. Writing

$$\delta^4\left(\sum\pi_r\overline{\pi}_r\right) \equiv \prod_{\dot{a},b=1}^{2}\delta\left(\sum_{r=1}^{3}\pi_r^b\overline{\pi}_{r\dot{a}}\right) \tag{3.7}$$

yields for the amplitude

$$\begin{aligned}\mathscr{A}_{-++}^{\text{tree}} &= \delta^4\left(\sum\pi_r\overline{\pi}_r\right)[2,3]\frac{\left(\pi_1^1\right)^3}{\pi_2^1\pi_3^1}f^{A_1A_2A_3}\int\delta\left(\pi_1^2-\pi_1^1\frac{\lambda^2}{\lambda^1}\right)\frac{\mathrm{d}\lambda^2}{\lambda^1}\\ &= \delta^4\left(\sum\pi_r\overline{\pi}_r\right)[2,3]\frac{\left(\pi_1^1\right)^2}{\pi_2^1\pi_3^1}f^{A_1A_2A_3}\\ &= \delta^4\left(\sum\pi_r\overline{\pi}_r\right)\frac{[2,3]^4}{[1,2]\,[2,3]\,[3,1]}f^{A_1A_2A_3}.\end{aligned}$$

The last equality holds since

$$0=\sum_{r=1}^{3}\pi_r^b\overline{\pi}_{r\dot{a}}\Longrightarrow\forall s\in\{1,2,3\}:\ 0=\sum_{r\neq s}[r,s]\,\pi_r^b,$$

where we contracted the expression with $\overline{\pi}_s^{\dot{a}}$ and used that $[s,s]=0$. For $n=3$ the relations can be written as

$$\frac{\pi_1^1}{\pi_2^1}=\frac{[2,3]}{[3,1]}\qquad\frac{\pi_1^1}{\pi_3^1}=\frac{[2,3]}{[1,2]}.$$

3.3.2 Degree one

For $d=1$, the range of integration includes besides the zero modes Z_0^I also the terms zZ_{-1}^I, which – due to the new factor e^{q_0} and $e^{q_0}Z_{-1}^I=Z_0^Ie^{q_0}$ (cf. eq. (3.3)) – does not annihilate the vacuum any more (cf. eqs. (3.2) and (3.4)). The vacuum expectation value of the currents $\langle 0|\,J^{A_1}(z_1)\ldots J^{A_n}(z_n)\,|0\rangle$ is of the form

$$\frac{f^{A_1\ldots A_n}}{(z_1-z_2)(z_2-z_3)\ldots(z_n-z_1)}.$$

Inserting the findings from the last section, we find the amplitude

$$\begin{aligned}\mathscr{A}^{\text{tree}}_{--+\ldots+}=\int\prod_{r=1}^{n}\frac{\mathrm{d}k_r}{k_r}\prod_{a=1}^{2}\delta\left(\pi_r^a-k_r\lambda^a(z)\right)\prod_{\dot{a}=1}^{2}\exp\left(\mathrm{i}k_r\mu^{\dot{a}}(z_r)\overline{\pi}_{r\dot{a}}\right)\\ \times k_1^4k_2^4\,\langle 0|\,e^{q_0}\psi^1(z_1)\psi^2(z_1)\psi^3(z_1)\psi^4(z_1)\psi^1(z_2)\psi^2(z_2)\psi^3(z_2)\psi^4(z_2)\,|0\rangle\\ \times\prod_{a,\dot{a}=1}^{2}\mathrm{d}^2\lambda^a\mathrm{d}^2\mu^{\dot{a}}\frac{f^{A_1\ldots A_n}\mathrm{d}z_1\ldots\mathrm{d}z_n}{(z_1-z_2)(z_2-z_3)\ldots(z_n-z_1)}\Big/\mathrm{d}\gamma_M\mathrm{d}\gamma_S.\end{aligned}$$

Since the modes of $Z^I(z)$ for $I\in\{5,6,7,8\}$ are anticommuting and thus $\psi_i^b\psi_i^b=0$ we have

$$\begin{aligned}\langle 0|\,e^{q_0}\psi^i(z_1)\psi^i(z_2)\,|0\rangle&=\langle 0|\,e^{q_0}z_1\psi_{-1}^i\psi_0^i\,|0\rangle+\langle 0|\,e^{q_0}z_2\psi_0^i\psi_{-1}^i\,|0\rangle\\&=(z_1-z_2)\,\langle 0|\,\psi_{-1}^i\psi_0^i\,|0\rangle=(z_1-z_2),\end{aligned}\tag{3.8}$$

where we used the normalisation $\langle 0|\,e^{dq_0}\psi_{-d}^i\cdots\psi_0^i\,|0\rangle=1$ (cf. [DG07] eq.(3.23)).

As in the case of $d=0$, we use the n δ-functions $\delta\left(\pi_r^1-k_r\lambda^1(z)\right)$ to perform the integration over the k's

$$\begin{aligned}\mathscr{A}^{\text{tree}}_{--+\ldots+}&=\int\prod_{r=1}^{n}\frac{\mathrm{d}k_r}{k_r}\prod_{a=1}^{2}\delta\left(\pi_r^a-k_r\lambda^a(z)\right)\prod_{\dot{a}=1}^{2}\delta\left(\sum_{r=1}^{n}k_r\overline{\pi}_{r\dot{a}}\right)\delta\left(\sum_{r=1}^{n}k_rz_r\overline{\pi}_{r\dot{a}}\right)\\&\quad\times k_1^4k_2^4(z_1-z_2)^4\prod_{a=1}^{2}\mathrm{d}^2\lambda^a\frac{f^{A_1\ldots A_n}\mathrm{d}z_1\ldots\mathrm{d}z_n}{(z_1-z_2)(z_2-z_3)\ldots(z_n-z_1)}\Big/\mathrm{d}\gamma_M\mathrm{d}\gamma_S\\&=\int\prod_{r=1}^{n}\frac{1}{\pi_r^1}\delta\left(\pi_r^2-\frac{\lambda^2(z_r)}{\lambda^1(z_r)}\pi_r^1\right)\prod_{\dot{a}=1}^{2}\delta\left(\sum_{r=1}^{n}\frac{\pi_r^1\overline{\pi}_{r\dot{a}}}{\lambda^1(z_r)}\right)\delta\left(\sum_{r=1}^{n}\frac{z_r\pi_r^1\overline{\pi}_{r\dot{a}}}{\lambda^1(z_r)}\right)\prod_{a=1}^{2}\mathrm{d}^2\lambda^a\\&\quad\times\left(\frac{\pi_1^1\pi_2^1(z_1-z_2)}{\lambda^1(z_1)\lambda^1(z_2)}\right)^4\frac{f^{A_1\ldots A_n}\mathrm{d}z_1\ldots\mathrm{d}z_n}{(z_1-z_2)(z_2-z_3)\ldots(z_n-z_1)}\Big/\mathrm{d}\gamma_M\mathrm{d}\gamma_S.\end{aligned}$$

With the first δ-function, i.e. $\pi_r^2 - \frac{\lambda^2(z_r)}{\lambda^1(z_r)}\pi_r^1 = 0$, we have for all $\dot{a}, b$

$$\sum_{r=1}^{n} \pi_r^b \overline{\pi}_{r\dot{a}} = \sum_{r=1}^{n} \frac{\lambda^b(z_r)\pi_r^1\overline{\pi}_{r\dot{a}}}{\lambda^1(z_r)} = \lambda_0^b \sum_{r=1}^{n} \frac{\pi_r^1\overline{\pi}_{r\dot{a}}}{\lambda^1(z_r)} + \lambda_{-1}^b \sum_{r=1}^{n} \frac{z_r\pi_r^1\overline{\pi}_{r\dot{a}}}{\lambda^1(z_r)}.$$

By bringing the four δ-functions inside the amplitude

$$\prod_{\dot{a}=1}^{2} \delta\left(\sum_{r=1}^{n} \frac{\pi_r^1\overline{\pi}_{r\dot{a}}}{\lambda^1(z_r)}\right) \delta\left(\sum_{r=1}^{n} \frac{z_r\pi_r^1\overline{\pi}_{r\dot{a}}}{\lambda^1(z_r)}\right)$$

into the desired momentum conservation δ-function introduced in eq. (3.7), we need to perform the transformation

$$\begin{pmatrix} \sum_{r=1}^{n} \frac{\pi_r^1\overline{\pi}_{r\dot{a}}}{\lambda^1(z_r)} \\ \sum_{r=1}^{n} \frac{z_r\pi_r^1\overline{\pi}_{r\dot{a}}}{\lambda^1(z_r)} \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda_0^1 & \lambda_{-1}^1 \\ \lambda_0^2 & \lambda_{-1}^2 \end{pmatrix} \begin{pmatrix} \sum_{r=1}^{n} \frac{\pi_r^1\overline{\pi}_{r\dot{a}}}{\lambda^1(z_r)} \\ \sum_{r=1}^{n} \frac{z_r\pi_r^1\overline{\pi}_{r\dot{a}}}{\lambda^1(z_r)} \end{pmatrix}$$

for both $\dot{a} = 1, 2$. Therefore, we may replace the four δ-functions by

$$\left(\lambda_0^1\lambda_{-1}^2 - \lambda_{-1}^1\lambda_0^2\right)^2 \delta^4\left(\sum \pi_r\overline{\pi}_r\right). \tag{3.9}$$

We introduce new coordinates

$$\zeta_r := \frac{\lambda^2(z_r)}{\lambda^1(z_r)} = \frac{\lambda_0^2 + \lambda_{-1}^2 z_r}{\lambda_0^1 + \lambda_{-1}^1 z_r} \tag{3.10}$$

for which we have

$$\begin{aligned} \zeta_i - \zeta_j = \frac{\lambda^2(z_i)}{\lambda^1(z_i)} - \frac{\lambda^2(z_j)}{\lambda^1(z_j)} &= \frac{\lambda^2(z_i)\lambda^1(z_j) - \lambda^2(z_j)\lambda^1(z_i)}{\lambda^1(z_i)\lambda^1(z_j)} \\ &= \frac{\lambda_0^1\lambda_{-1}^2 - \lambda_0^2\lambda_{-1}^1}{\lambda^1(z_i)\lambda^1(z_j)}(z_i - z_j). \end{aligned} \tag{3.11}$$

The quartic factor in the amplitude thus becomes

$$\left(\frac{z_1 - z_2}{\lambda^1(z_1)\lambda^1(z_2)}\right)^4 = (\zeta_1 - \zeta_2)^4 \cdot \frac{1}{\left(\lambda_0^1\lambda_{-1}^2 - \lambda_0^2\lambda_{-1}^1\right)^4}.$$

Finally, we have to translate the differentials

$$\mathrm{d}\zeta_r = \frac{\mathrm{d}\zeta_r}{\mathrm{d}z_i}\mathrm{d}z_i = \frac{\lambda_{-1}\lambda_0^1 - \lambda_0^2\lambda_{-1}^1}{\left(\lambda^1(z_r)\right)^2}\mathrm{d}z_r$$

with

$$\frac{\mathrm{d}z_r}{z_r - z_{r+1}} = \frac{\left(\lambda^1(z_r)\right)^2}{\lambda_{-1}\lambda_0^1 - \lambda_0^2\lambda_{-1}^1} \frac{\mathrm{d}\zeta_r}{z_r - z_{r+1}} = \frac{\lambda^1(z_r)}{\lambda^1(z_{r+1})} \frac{\mathrm{d}\zeta_r}{\zeta_r - \zeta_{r+1}},$$

where we used eq. (3.11). Note that by multiplying over all $r \in \{1, \ldots, n\}$, the factors $\frac{\lambda^1(z_r)}{\lambda^1(z_{r+1})}$ cancel.

After all these transformations, we remain with the expression

$$\mathscr{A}^{\text{tree}}_{--+\ldots+} = \int \prod_{r=1}^{n} \frac{1}{\pi_r^1} \prod_{r=1}^{n} \delta\left(\pi_r^2 - \zeta_r \pi_r^1\right) \delta^4\left(\sum \pi_r \overline{\pi}_r\right) \frac{\mathrm{d}^2\lambda^1 \mathrm{d}^2\lambda^2}{\left(\lambda_0^1 \lambda_{-1}^2 - \lambda_0^2 \lambda_{-1}^1\right)^2} \times \left(\pi_1^1 \pi_2^1 (\zeta_1 - \zeta_2)\right)^4 \frac{f^{A_1 \ldots A_n} \mathrm{d}\zeta_1 \ldots \mathrm{d}\zeta_n}{(\zeta_1 - \zeta_2) \ldots (\zeta_n - \zeta_1)} \Big/ \mathrm{d}\gamma_M \mathrm{d}\gamma_S.$$

With the invariant measure on the product of the Möbius and scaling groups

$$\mathrm{d}\gamma_M \mathrm{d}\gamma_S = \frac{\mathrm{d}^2\lambda^1 \mathrm{d}^2\lambda^2}{\left(\lambda_0^1 \lambda_{-1}^2 - \lambda_{-1}^1 \lambda_0^2\right)^2}$$

we can perform the final integration, yielding the Parke–Taylor formula (cf. eq. (2.9))

$$\begin{aligned}
\mathscr{A}^{\text{tree}}_{--+\ldots+} &= \int \prod_{r=1}^{n} \frac{1}{\left(\pi_r^1\right)^2} \delta\left(\frac{\pi_r^2}{\pi_r^1} - \zeta_r\right) \delta^4\left(\sum \pi_r \overline{\pi}_r\right) \left(\pi_1^1 \pi_2^1 (\zeta_1 - \zeta_2)\right)^4 \frac{f^{A_1 \ldots A_n} \mathrm{d}\zeta_1 \ldots \mathrm{d}\zeta_n}{(\zeta_1 - \zeta_2) \ldots (\zeta_n - \zeta_1)} \qquad (3.12) \\
&= \delta^4\left(\sum \pi_r \overline{\pi}_r\right) \left(\pi_1^2 \pi_2^1 - \pi_2^2 \pi_1^1\right)^4 f^{A_1 \ldots A_n} \prod_{r=1}^{n} \frac{1}{\left(\pi_r^1\right)^2} \frac{1}{\frac{\pi_r^2}{\pi_r^1} - \frac{\pi_{r+1}^2}{\pi_{r+1}^1}} \\
&= \delta^4\left(\sum \pi_r \overline{\pi}_r\right) \left(\pi_1^2 \pi_2^1 - \pi_2^2 \pi_1^1\right)^4 f^{A_1 \ldots A_n} \prod_{r=1}^{n} \frac{1}{\pi_r^2 \pi_{r+1}^1 - \pi_{r+1}^2 \pi_r^1} \\
&= \delta^4\left(\sum \pi_r \overline{\pi}_r\right) \frac{\langle 1,2 \rangle^4}{\langle 1,2 \rangle \langle 2,3 \rangle \ldots \langle n,1 \rangle} f^{A_1 \ldots A_n},
\end{aligned}$$

where we denoted $\pi_{n+1}^a \equiv \pi_1^a$.

Chapter 4

Selberg integrals

The motivation for investigating Selberg integrals and its limits stems from the findings of Mafra, Schlotterer, and Stieberger in [MSS13] that the $(L+1)$-point tree amplitudes for open strings split into contributions from SYM theory and string corrections F. Following the review in [BK19], the splitting amounts to

$$\mathscr{A}^{\text{tree}}_{\text{open}}(1,L,L-1,\ldots 2,L+1) = \sum_{\sigma\in S_{L-2}} F^{\sigma}\cdot\mathscr{A}^{\text{SYM}}_{L+1}(1,\sigma(L,L-1,\ldots,3),2,L+1)\,.$$

The string corrections F are kinematical functions and can be expressed in terms of the Mandelstam variables. They are given by

$$\hat{F}^{\sigma}(\alpha') = (-1)^L \int_{\mathscr{C}} \prod_{i=3}^{L} \mathrm{d}x_i \ \mathrm{KN}\ \sigma\left[\prod_{k=3}^{L}\left(\sum_{j=k+1}^{L}\frac{s_{jk}}{x_{jk}}+\frac{s_{k1}}{x_{k1}}\right)\right],$$

where the permutation σ acts on the $L-2$ indices inside the square brackets and the domain of integration $\mathscr{C}$ amounts to $0=x_1\leq x_i<x_{i+1}<x_2=1$. The Koba–Nielsen factor KB is given by

$$\mathrm{KB} = \prod_{\substack{1\leq i,j\leq N:\\ x_1\leq x_i<x_j\leq x_2}} \left|x_i-x_j\right|^{s_{ij}}.$$

These string corrections F are Selberg integrals and their evaluation shall be recapitulated in this section. It turns out that including an extra point x_3 into the integral and considering its limits $x_3\to 0,1$ reveals further features of this class of integrals and allows for recursion relations connecting the N- and $(N-1)$-point amplitudes via the Drinfeld associator. While the original articles to the topic are [Bro+14; Kad20], in the derivation of both the Selberg integral and the Drinfeld associator, we will follow the lines of [BK19].

4.1 Iterated integrals and integrals of SELBERG type

We begin by introducing the so-called Goncharov polylogarithms defined by the iterative sequence

$$G(a_1,\dots,a_r;x) := \int_0^1 \mathrm{d}t \frac{1}{t-a_1} G(a_2,\dots,a_r;x), \quad G(;x) := 1.$$

To first obtain a better understanding of these iterated integrals, we want to restrict the coefficients a_i to 0 and 1. With the exclusion of the divergence $\int_0^1 \mathrm{d}t_n \frac{1}{t} \to \infty$ in the innermost integral by restricting $a_r = 1$, we find an integral representation of the *multiple polylogarithms* as the lemma below shows. These are denoted by $G_w(x)$ with w being a word of the form

$$w = e_0^{n_r-1} e_1 e_0^{n_{r-1}-1} \dots e_0^{n_1-1} e_1, \qquad n_i \geq 1$$

and form a subclass of the Goncharov polylogarithms

$$G_w(x) = G\big(\underbrace{0,\dots,0}_{n_r-1},1,\dots,\underbrace{0,\dots,0}_{n_1-1},1;x\big).$$

Lemma 4.1.1. *Up to a sign, the above expression for G_w coincides with the sum representation of the multiple polylogarithms* $\mathrm{Li}_{n_1\dots n_r}$

$$G_w(x) = (-1)^r \sum_{1\leq k_1<\dots<k_r} \frac{x^{k_r}}{k_1^{n_1}\dots k_r^{n_r}} = (-1)^r \mathrm{Li}_{n_1\dots n_r}(x). \tag{4.1}$$

Proof. We begin by showing the statement for $r=1$. Then, for $|x|<1$

$$\begin{aligned}
G(\underbrace{0,\dots,0}_{n-1};x) &= \int_0^x \frac{\mathrm{d}t_1}{t_1} G(\underbrace{0,\dots,0}_{n-2};t_1) = \int_0^x \frac{\mathrm{d}t_1}{t_1}\int_0^{t_1}\frac{\mathrm{d}t_2}{t_2}\dots\int_0^{t_{n-1}}\frac{\mathrm{d}t_n}{t_n-1} \\
&= -\sum_{m\in\mathbb{N}_0}\int_0^x \frac{\mathrm{d}t_1}{t_1}\int_0^{t_1}\frac{\mathrm{d}t_2}{t_2}\dots\int_0^{t_{n-1}} \mathrm{d}t_n\,(t_n)^m \\
&= -\sum_{m\in\mathbb{N}_0}\frac{1}{(m+1)}\int_0^x \frac{\mathrm{d}t_1}{t_1}\int_0^{t_1}\frac{\mathrm{d}t_2}{t_2}\dots\int_0^{t_{n-2}}\frac{\mathrm{d}t_{n-1}}{t_{n-1}}(t_{n-1})^{m+1} \\
&= -\sum_{m\in\mathbb{N}_0}\frac{1}{(m+1)^{n-1}}\int_0^x\frac{\mathrm{d}t_1}{t_1}(t_1)^{m+1} \\
&= -\sum_{m\in\mathbb{N}}\frac{x^m}{m^n}.
\end{aligned}$$

Assuming that the formula is correct for r, we have

$$
\begin{aligned}
&G(\underbrace{0,\dots,0}_{n_{r+1}-1},1,\dots,\underbrace{0,\dots,0}_{n_1-1},1;x) = \\
&\qquad = \int_0^x \frac{\mathrm{d}t_1}{t_1}\dots\int_0^{t_{n_{r+1}-1}} \frac{\mathrm{d}t_{n_{r+1}}}{t_{n_{r+1}}-1} G(\underbrace{0,\dots,0}_{n_r-1},1,\dots,\underbrace{0,\dots,0}_{n_1-1},1;t_{n_{r+1}}) \\
&\qquad = -(-1)^r \sum_{1\le k_1<\dots k_r} \frac{1}{k_1^{n_1}\dots k_r^{n_r}} \sum_{m\in\mathbb{N}_0} \int_0^x \frac{\mathrm{d}t_1}{t_1}\dots\int_0^{t_{n_{r+1}-1}} \mathrm{d}t_{n_{r+1}} \left(t_{n_{r+1}}\right)^{k_r+m} \\
&\qquad = (-1)^{r+1} \sum_{1\le k_1<\dots k_r} \frac{1}{k_1^{n_1}\dots k_r^{n_r}} \sum_{m\in\mathbb{N}_0} \frac{1}{(k_r+m+1)^{n_{r+1}-1}} \int_0^x \mathrm{d}t_1\,(t_1)^{k_r+m} \\
&\qquad = (-1)^{r+1} \sum_{1\le k_1<\dots k_{r+1}} \frac{x^{k_{r+1}}}{k_1^{n_1}\dots k_{r+1}^{n_{r+1}}},
\end{aligned}
$$

where we substituted m for the summation variable $k_{r+1} = k_r + m > k_r$. This finishes the proof by induction. □

From both the definition in section 4.1 and the form in eq. (4.1) it is clear that the multiple polylogarithms diverge for words beginning with e_1 (which amounts to $n_r = 1$) as well. The regularisation of these integrals is achieved by reducing the domain of integration to $[\epsilon, 1-\epsilon]$, where $\epsilon > 0$ is small. The part in the ϵ expansion independent of ϵ is then taken to be the regularised Goncharov polylogarithm.

Even though we restricted the words to end in e_1 in the original definition of the multiple polylogarithms, there is a natural generalisation to all word w. For two words $w' = i_1 \dots i_r$ and $w'' = i_{r+1} \dots i_{r+s}$ we define the so-called *shuffle product* by

$$
w' \sqcup\!\sqcup\, w'' = \sum_{\sigma\in\Sigma(r,s)} i_{\sigma(1)}\dots i_{\sigma(r+s)},
$$

where $\Sigma(r,s)$ denotes the subset of S_{r+1} that leaves the underlying ordering of the words untouched, i.e.

$$
\sigma(i) < \sigma(j) \quad \Leftrightarrow \quad i < j \wedge \left(i,j \in \{1,\dots,r\} \vee i,j \in \{r+1,\dots,r+s\}\right).
$$

Intuitively, this is equivalent to the riffle shuffling of two decks of cards. The product of two multiple polylogarithms is equal to the one corresponding to the shuffle product of the words

$$
G_{w'} G_{w''} = G_{w' \sqcup\!\sqcup\, w''}.
$$

Then, the definition

$$
G_{e_0^n}(x) := \frac{\log(x)^n}{n!}
$$

gives rise to multiple polylogarithms $G_w(x)$ where w is *any* word in e_0 and e_1, including those ending in e_0. Additionally, they obey the differential equation

$$dG_{e_i w}(x) = w_i G_w(x), \quad \text{where} \quad w_0 = \frac{\mathrm{d}x}{x},\ w_1 = \frac{\mathrm{d}x}{x-1},$$

which follows immediately from the definition in section 4.1.

We introduce L real variables

$$0 = x_1 < x_L < x_{L-1} < \ldots < x_2 = 1$$

which later will act as the insertion points of vertex operators. The Selberg integral is defined by

$$S[i_{k+1},\ldots,i_L](x_1,\ldots,x_k) = \int_0^{x_k} \frac{\mathrm{d}x_{k+1}}{x_{k+1} - x_{i_{k+1}}} S[i_{k+2},\ldots,i_L](x_1,\ldots,x_{k+1}), \tag{4.2}$$

with $1 \le i_p < p$ for this expression to be well defined. The core of this iteration is called the Selberg seed and is given by the Koba–Nielsen factor

$$S \equiv S[](x_1,\ldots,x_L) := \exp\left(\sum_{0\le x_i<x_j\le 1} s_{ij} \log(x_j - x_i)\right) = \prod_{0\le x_i<x_j\le 1} |x_i - x_j|^{s_{ij}}. \tag{4.3}$$

4.2 KNIZHNIK–ZAMOLODCHIKOV equation

To make the following more legible, we write for the differences $x_{ij} \equiv x_i - x_j$. As will become clear later on when we introduce the Drinfeld associator, we are interested in a type of Selberg integral of the form $S[i_4,\ldots i_L](0,1,x_3)$ and its limits $x_3 \to \{0,1\}$. To determine these, we would like to find an expression for

$$\frac{\mathrm{d}}{\mathrm{d}x_3} S[i_4,\ldots,i_L](0,1,x_3) = \frac{\mathrm{d}}{\mathrm{d}x_3} \int_{\mathscr{C}(x_3)} \prod_{j=4}^{L} \mathrm{d}x_j S \prod_{k=4}^{L} \frac{1}{x_{k,i_k}}, \tag{4.4}$$

where the domain of integration is given by

$$\mathscr{C}(x_3) = \{(x_4,\ldots,x_L) \mid 0 = x_1 < x_L < x_{L-1} < \ldots < x_3 < x_2 = 1\}.$$

Since the integrand vanishes for $x_4 = x_3$, the term arising from the derivative of the domain of integration can be omitted [1]. Using partial integration and the fact that

$$\frac{\partial}{\partial x_i}\frac{1}{x_{ij}} = -\frac{\partial}{\partial x_j}\frac{1}{x_{ij}}$$

[1] In the case of $i_4 = 3$, one should regard the resulting Selberg integral as a linear combination of basis elements

$$\mathscr{B}_{i'_4,\ldots i'_L} = \{S[i_4,\ldots i_L](0,1,x_3) \mid 1 \le i_k < k, i_k \ne i'_k\},$$

of a basis with $i'_4 = 3$. The fact that this is indeed a basis can be seen using partial fractioning.

we may rewrite eq. (4.4) as

$$\frac{\mathrm{d}}{\mathrm{d}x_3} S[i_4,\dots,i_L](0,1,x_3) = \int_{\mathscr{C}(x_3)} \prod_{j=4}^{L} \mathrm{d}x_j \left(\sum_{j\in U_3} \frac{\partial}{\partial x_j} S \right) \prod_{k=4}^{L} \frac{1}{x_{k,i_k}},$$

where the set U_3 is given by

$$U_3 = \left\{ j \in \{3,\dots,L\} \;\middle|\; j = 3 \text{ or there exist labels } 3 = j_1, j_2, \dots j_m = j \text{ where } \prod_{i=1}^{m-1} \frac{1}{x_{j_{i+1},j_i}} \text{ is a factor of } \prod_{k=4}^{L} \frac{1}{x_{k i_k}} \right\}. \tag{4.5}$$

From the definition of the Selberg seed in eq. (4.3) it is clear that the derivatives on it act as

$$\frac{\partial}{\partial x_j} S = \sum_{l\neq j} \frac{s_{jl}}{x_{jl}} S. \tag{4.6}$$

Thus, the final form of the derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}x_3} S[i_4,\dots,i_L](0,1,x_3) = \int_{\mathscr{C}(x_3)} \prod_{j=4}^{L} \mathrm{d}x_j \left(\sum_{j\in U_3} \sum_{l\neq U_3} \frac{s_{jl}}{x_{jl}} S \right) \prod_{k=4}^{L} \frac{1}{x_{k,i_k}}. \tag{4.7}$$

Repeated application of partial fractioning will yield a sum of Selberg integrals with different indices i_j. Finally, all terms will be proportional to either

$$\frac{1}{x_{31}} = \frac{1}{x_3} \quad \text{or} \quad \frac{1}{x_{32}} = \frac{1}{x_3 - 1}.$$

since these denominator cannot be simplified further by partial fractioning. Denoting the vector of all admissible indices i_j by

$$\mathbf{S}(x_3) \equiv (S[i_4,\dots,i_L](0,1,x_3))_{1\leq i_k<k}, \tag{4.8}$$

consisting of $(L-1)\cdot\ldots\cdot 3 = (L-1)!/2$ entries, we can summarise the result in a so-called Knizhnik—Zamolodchikov (KZ) differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x_3} \mathbf{S}(x_3) = \left(\frac{e_0}{x_3} + \frac{e_1}{x_3 - 1} \right) \mathbf{S}(x_3), \tag{4.9}$$

with e_0 and e_1 being square matrices of size $(L-1)!/2 \times (L-1)!/2$. Using partial fractioning and integration by parts, we may exclude one particular index i'_k for each $4 \leq k \leq L$. Thus, for each index family $\{i'_k\}_{1\leq k\leq N}$ we find a basis

$$\mathscr{B}_{i'_4,\dots,i'_L} := \left\{ S[i_4,\dots,i_L](0,1,x_3) \;\middle|\; 1 \leq i_k < k,\ i_k \neq i'_k \right\}. \tag{4.10}$$

This possible exclusion of a specific index will become useful in section 4.4, where we will investigate the limits $x_3 \to x_1, x_2$.

4.3 DRINFELD associator

We introduce a representation ρ of a Lie algebra with generators e_0 and e_1. For readability, we abbreviate $\rho(e_i) \equiv e_i$. Let $F(x)$ be a function acted upon by the two Lie algebra generators which satisfies the KZ equation

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x) = \left(\frac{e_0}{x} + \frac{e_1}{x-1}\right)F(x).$$

One solution to this differential equation is given by

$$L(x) = \sum_{w \in \{e_0,e_1\}^\times} w G_w(x),$$

where $\{e_0, e_1\}^\times$ denotes the set of all words consisting of e_0 and e_1 (including the empty word). By writing $w = e_{i_1} \dots e_{i_n}$ we actually mean $\rho\left(e_{i_1}\right) \dots \rho\left(e_{i_n}\right)$. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}L(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\Big[\sum_{w \in \{e_0,e_1\}^\times} \left(e_0 w G_{e_0 w}(x) + e_1 w G_{e_1 w}(x)\right) + 1\Big] \\
&= \sum_{w \in \{e_0,e_1\}^\times} \left(\frac{e_0 w}{x} G_w(x) + \frac{e_1 w}{x-1} G_w(x)\right) \\
&= \left(\frac{e_0}{x} + \frac{e_1}{x-1}\right)L(x).
\end{aligned}$$

When considering the limit $x \to 0$, eq. (4.1) implies that the terms corresponding to words ending in e_1 vanish. We may write the summation variable w as the shuffle product $w = w' \sqcup\!\sqcup\, w''$, where w' consist solely of e_0s and w'' of e_1s. Then,

$$\begin{aligned}
\lim_{x\to 0} L(x) &= \lim_{x\to 0} \sum_{w',w''} w' \sqcup\!\sqcup\, w'' G_{w' \sqcup\!\sqcup w''}(x) \\
&= \lim_{x\to 0} \sum_{w',w''} w' \sqcup\!\sqcup\, w'' G_{w'}(x) G_{w''}(x) \\
&= \lim_{x\to 0} \sum_{w''} G_{w''}(x) \sum_{n=0}^{\infty} e_0^n \sqcup\!\sqcup\, w'' \frac{\log(x)^n}{n!}.
\end{aligned}$$

The dominant term in this series corresponds to w'' being the empty word. For $w'' = e_1^k$ and $k > 0$, the polylogarithms $G_{w''}(x)$ contribute a positive power of x and therefore take no part in the limit $x \to 0$. Thus, we have

$$L(x) \sim x^{e_0}, \qquad \text{as } x \to 0.$$

Since the KZ equation written as $DF(x) = 0$ exhibits the symmetry $x \mapsto \hat{x} = 1 - x$, $\rho \mapsto \hat{\rho}$, where $\hat{\rho}(e_0) = \rho(e_1)$ and vice versa,

$$\begin{aligned}
\hat{D} &= \frac{\mathrm{d}}{\mathrm{d}(1-x)} - \frac{e_1}{1-x} - \frac{e_0}{1-x-1} \\
&= -\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{e_1}{x-1} - \frac{e_0}{x}\right) \\
&= -D,
\end{aligned}$$

we find another solution $L_1(x)$ with the asymptotic behaviour

$$L_1(x) \sim (1-x)^{e_1}, \qquad \text{as } x \to 1.$$

This new solution $L_1(x)$ is given by $L(1-x)$, where the *representations* $\rho(e_0)$ and $\rho(e_1)$ are interchanged. Note that the algebraic influence of the Lie algebra generators on the multiple polylogarithms G_w remains invariant.

Let us now take a step back and consider a general solution $F(z)$ of the KZ equation. We may now consider regularised boundary values as

$$C_0 := \lim_{x\to 0} x^{-e_0} F(x), \qquad C_1 := \lim_{x\to 1} (1-x)^{-x_0} F(x).$$

For any two functions $\alpha(x)$ and $\beta(x)$ satisfying the KZ equation, the product $(\alpha\beta^{-1})(x)$ is independent of x because

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\alpha}{\beta}(x) &= \frac{\alpha'}{\beta}(x) - \frac{\alpha\beta'}{\beta^2}(x)\\ &= \frac{\alpha\alpha'}{\alpha\beta}(x) - \frac{\alpha\beta'}{\beta^2}(x)\\ &= D\left(\frac{\alpha}{\beta} - \frac{\alpha}{\beta}\right) = 0.\end{aligned}$$

In particular, the products $(L_1(x))^{-1}L(x)$ and $(L_1(x))^{-1}F(x)$ are invariant of x. Thus,

$$\begin{aligned}(L_1(x))^{-1}L(x)C_0 &= \lim_{x\to 0}(L_1(x))^{-1}L(x)C_0\\ &= \lim_{x\to 0}(L_1(x))^{-1}x^{e_0}x^{-e_0}F(x)\\ &= \lim_{x\to 0}(L_1(x))^{-1}F(x)\\ &= \lim_{x\to 1}(L_1(x))^{-1}F(x)\\ &= C_1.\end{aligned}$$

This proves that the Drinfeld associator defined as

$$\Phi(e_0,e_1) := (L_1(x))^{-1}L(x)$$

maps the regularised boundary value C_0 to C_1

$$C_1 = \Phi(e_0,e_1)C_0, \tag{4.11}$$

4.4 Regularised boundary values of SELBERG integrals

The Drinfeld associator introduced in the previous subsection allows us to relate the two regularised boundary values

$$\mathbf{C}_0 = \lim_{x_3\to 0} x_3^{-e_0}\mathbf{S}(x_3), \qquad \mathbf{C}_1 = \lim_{x_3\to 1}(1-x_3)^{-e_1}\mathbf{S}(x_3)$$

to each other. It turns out that these two limits represent string tree amplitudes with $L-1$ and L external particles, respectively, implying that the Drinfeld associator takes L-particle amplitudes and generates a $L+1$-particle. The regularising factors $x_3^{-e_0}$ and $(1-x_3)^{-e_1}$ diverging as $x_3 \to 0, 1$, respectively, are necessary, since (as we discussed in section 4.2) the Selberg integrals vanish for $x_i \to x_j$.

Boundary value $\mathbf{C}_1$

Since we are interested in the limit $x_3 \to x_2 = 1$, we may restrict ourselves to the basis elements in $\mathscr{B}_{2,\ldots,2} \cap \mathscr{B}_{3,\ldots,3}$ (cf. eq. (4.10)). For these elements, the differentiation with respect to x_3 acts solely on the Selberg seed and yields a factor $\frac{s_{23}}{x_{23}}$. Thus the eigenvalue of e_1 acting on the integrals is simply s_{23}. We have

$$
\begin{aligned}
&\lim_{x_3 \to 1} (1-x_3)^{-e_1} S\,[i_4, \ldots i_L]\,(0,1,x_3) \\
&\qquad = \lim_{x_3 \to x_2} x_{23}^{-s_{23}}\,[i_4, \ldots i_L]\,(0,1,x_3) \\
&\qquad = \lim_{x_3 \to x_2} \int_{\mathscr{C}(x_3)} \prod_{i=4}^{L} \mathrm{d}x_i \prod_{0 \le x_j < x_k < x_3} x_{kj}^{s_{kj}} \prod_{0 \le x_l < x_3} x_{3l}^{s_{3l}} \prod_{0 \le x_m < x_3} x_{2m}^{s_{2m}} \prod_{n=4}^{L} \frac{1}{x_{ni_n}} \\
&\qquad = \int_{\mathscr{C}(x_2)} \prod_{i=4}^{L} \mathrm{d}x_i \prod_{0 \le x_j < x_k < x_3} x_{kj}^{s_{kj}} \prod_{0 \le x_m < x_3} x_{2m}^{s_{2m}+s_{3m}} \prod_{n=4}^{L} \frac{1}{x_{ni_n}} \\
&\qquad = S\,[i_4, \ldots {}_L]\,(0,1,x_3 = x_2) \Big|_{s_{23}=0}^{s_{2n} \to s_{2n}+s_{3n}},
\end{aligned}
$$

where the regularisation factor $x_{23}^{-s_{23}}$ canceled the vanishing expression $x_{23}^{s_{23}}$ in the Selberg seed. We see that the momentum of the state x_3 was absorbed by x_2, yielding the resulting Mandelstam variables $s_{2n} \to \tilde{s}_{2n} = s_{2n} + s_{3n}$. Thus, the new set of integrals describes L-point amplitudes with the insertion points $x_1, x_2, x_4, \ldots x_{L+1}$.

Boundary value $\mathbf{C}_0$

As for the limit $x_3 \to x_2 = 1$, we will consider the basis elements $\mathscr{B}_{2,\ldots,2}$. It follows from eq. (4.7) and the KZ equation that the matrix e_0 is a linear combination of the Mandelstam variables s_{ij} with $i, j \neq 2, L+1$. The variables s_{2n} are missing, since terms that include a factor of x_{2n}^{-1} must be part of the e_1/x_{13} term in the KZ equation (after a sufficient amount of application of partial fractioning). If they would be generated by the matrix e_0, the Selberg integrals on the r.h.s. of the KZ equation would contain factors x_{2n}^{-1}, which is prohibited by the choice of basis. Let us denote an admissible eigenvalue of e_0 by $\tilde{s}$. Then,

$$
\begin{aligned}
&\lim_{x_3 \to 0} x_{31}^{-\tilde{s}} S\,[i_4, \ldots, i_L]\,(0,1,x_3) \\
&\qquad = \lim_{x_3 \to 0} x_3^{-\tilde{s}} \int_{\mathscr{C}(x_3)} \prod_{i=4}^{L} \mathrm{d}x_i \prod_{0 \le x_j < x_k \le 1} x_{kj}^{s_{jk}} \prod_{0 \le x_l < x_3} x_{3l}^{s_{3l}} \prod_{0 \le x_m < x_2} x_{2m}^{s_{2m}} \prod_{n=4}^{L} \frac{1}{x_{ni_n}}.
\end{aligned}
$$

We note that the regularisation factor $x_3^{-\tilde{s}}$ must cancel the factor

$$x_3^{-\tilde{s}} \sim \prod_{0\le x_j<x_k\le 1} x_{kj}^{s_{jk}} \prod_{0\le x_l<x_3} x_{3l}^{s_{3l}}$$

for the boundary value $0 \le x_i < x_3 \to 0$ to be non-zero. This is true for the integrals with $\tilde{s}_0 = s_{1,3,4,\ldots L}$ (cf. eq. (2.6)). We perform a change of variables $\omega_i := x_i/x_3$, yielding

$$\begin{aligned}
&\lim_{x_3\to 0} x_{31}^{-\tilde{s}_0} S\,[i_4,\ldots,i_L]\,(0,1,x_3) \\
&\quad= \lim_{x_3\to 0} x_3^{-\tilde{s}_0} \int_{0=\omega_1<\omega_i<\omega_3=1} \prod_{i=4}^{L} \mathrm{d}\omega_i x_3 \prod_{0\le\omega_j<\omega_k\le 1} x_3^{s_{jk}} \omega_{kj}^{s_{jk}} \prod_{0\le\omega_l<\omega_3} x_3^{s_{3l}} \omega_{3l}^{s_{3l}} \prod_{0\le x_m<x_2} \\
&\qquad \times (1-x_3\omega_m)^{s_{2m}} \prod_{n=4}^{L} \frac{1}{x_3\omega_{ni_n}} \\
&\quad= \int_{0=\omega_1<\omega_i<\omega_3=1} \prod_{i=4}^{L} \mathrm{d}\omega_i \prod_{0\le\omega_j<\omega_k\le 1} \omega_{kj}^{s_{jk}} \prod_{0\le\omega_l<\omega_3} \omega_{3l}^{s_{3l}} \prod_{n=4}^{L} \frac{1}{\omega_{ni_n}} \\
&\quad= S\,[i_4,\ldots i_L]\,(0,1,x_3=0)\Big|_{s_{2n}=0}.
\end{aligned}$$

Integrals in the KZ equation with an eigenvalue $\tilde{s} \neq \tilde{s}_0$ vanish due to the discussion above. Therefore, the entries of the regularised boundary value $\mathbf{C}_0$ are linear combinations of $S\,[i_4,\ldots i_L]\,(0,1,x_3=0)\big|_{s_{2n}=0}$ and 0. As in the limit $x_3 \to 1$, we are left with an L-point string amplitude. Nevertheless, the limit $x_3 \to 0$ amounts to removing all momentum from the external particle x_2. Since a particle with no momentum can hardly contribute to a scattering process, we are led to interpret the result as an $L-1$-point amplitude. Thus, the Drinfeld associator (connecting the boundary values $\mathbf{C_0}$ and $\mathbf{C_1}$) encompasses the insertion or removal of an external particle in the amplitude.

Chapter 5

Towards a twistor Selberg integral

Does the structure of the gluon scattering amplitudes discussed in chapter 3 allow them to be formulated as Selberg-like integrals? If so, can the corresponding Drinfeld associator be used to come up with a new recursion relation for these amplitudes? In the following, we list and elaborate on the motivations, findings, and problems that arose while looking for answers to these questions.

In the twistor string theory proposed by Berkovits in [Ber04], the n gluon tree-level scattering super-amplitudes of $N=4$ SYM theory are given by (cf. eq. (3.6))

$$\begin{aligned}\mathscr{A}_n^{\text{tree}} = \int \mathrm{d}^{2d+2}a\, \mathrm{d}^{2d+2}b\, \mathrm{d}^{4d+4}\gamma \int \prod_{i=1}^{n} \mathrm{d}z_i \left(\mathrm{Vol}(GL(2))\right)^{-1} \mathrm{Tr}\left[\prod_{l=1}^{n} \Phi_l\left(\frac{\psi^A(z_l)}{\lambda^1(z_l)}\right)\right] \\ \times \prod_{j=1}^{n-1} \left(z_j - z_{j+1}\right)^{-1} (z_n - z_1)^{-1} \prod_{k=1}^{n} \delta\left(\frac{\pi_r^2}{\pi_r^1} - \frac{\lambda^2(z_r)}{\lambda^1(z_r)}\right) \exp\left(\mathrm{i}\bar{\pi}_r^{\dot{\alpha}} \pi_r^1 \frac{\mu_{\dot{\alpha}}(z_r)}{\lambda^1(z_r)}\right),\end{aligned} \tag{5.1}$$

where the δ-functions restrict the integrations over z_i to an algebraic curve of degree d. The z_i-dependent spinor variables are given by the expansions (cf. eq. (3.1))

$$\lambda^\alpha(z) = \sum_{i=0}^{d} a_{-i}^\alpha z^i, \quad \mu^{\dot{\alpha}}(z) = \sum_{i=0}^{d} b_{-i}^{\dot{\alpha}}, \quad \psi^A(z) = \sum_{i=0}^{d} \gamma_{-i}^A z^i.$$

The termination of the expansion is dictated by the annihilating property of the modes acting on the vacuum (cf. eqs. (3.2) and (3.3)). The amplitude concerning q negative and $n-q$ positive helicity gluons is recovered by setting $d = q-1$. In sections 3.3.1 and 3.3.2 we verified this result for $d=0$ and $d=1$. This amplitude makes use of Witten's conjecture [Wit04] that the external points in an N^k MHV tree-level scattering process described in twistor space must lie on curves of degree $k+1$ of genus zero. The underlying space of this curve is $\mathbb{C}^n \times \mathbb{CP}^3$, where the three (complex) degrees of freedom of $\mathbb{CP}^3$ are reduced to one by the incidence relation in eq. (2.8). Each of the $n+1$ copies of twistor space (n for the insertion points of external particles and one for the field $Z(z) = (\lambda(Z), \mu(z), \psi(z)))$ may be described with the restriction $\pi^1 = 1$ and $\lambda^1(z_i) = 1$, $\forall i$.

As we have shown explicitly for the MHV amplitude at the end of section 3.3.2, the selection of a split signature spacetime restricts the domain of integration to a subset of $\mathbb{R}$. For higher degrees, the $d+1$ δ-functions arising after integration of the exponential factor (where two of them are needed to enforce momentum conservation, cf. section 3.3.2) are expected to ensure this one-dimensional structure. This geometric property of the amplitude is one indicator for the resemblance between the gluon amplitudes and the Selberg integral where each integration is taken over a subset of $[0,1]$.

Each insertion of a new external particle gives rise to another integration over the complex plane restricted by a δ-function. Assuming the new particle's helicity to be negative, another eight integrations over the twistor modes $\mathrm{d}a^2$, $\mathrm{d}b^2$, and $\mathrm{d}\gamma^4$ appear. While the four fermionic mode integrations are solved by translating the fields $\psi(z)$ into z-variables (cf. eq. (3.8)), the integration of the exponential factor over the μ-modes b^2 yields two δ-functions, which ensure energy-momentum conservation (cf. the discussion leading to eq. (3.9)).

Our initial idea was that the δ-functions appearing in the amplitude above may take on the role of the denominator $(x_{k+1} - x_{i_{k+1}})^{-1}$ in the definition of the Selberg integral in eq. (4.2). The similar scaling behaviour $f(\alpha x) = x/a$ of both functions for $\alpha \in \mathbb{R}^+$ indicates that the properties of the Selberg integrals may be derived in an analogous fashion for the gluon amplitudes. Following chapter 4, the δ-functions should arise from a derivative of an exponential factor similar to the Selberg seed in eq. (4.3). The required algebraic form of this factor will be the topic of discussion of the following subsections. In order for the derivative to generate a δ-function, the exponential factor under consideration must contain a Heaviside-function $\theta(x)$ with at least one of the integration variables in its argument. In the same way that the Selberg seed contains the exponential of the antiderivative of x^{-1}, namely $\log(x)$, the Heaviside function should appear as the distributional antiderivative of $\delta(x)$

$$\theta'[\varphi] = \delta[\varphi].$$

The crucial feature of the Selberg integrals is the arising of poles following the differentiation with respect to the extra point x_3. Consequently, the vector of all admissible integrals satisfies the KZ equation, allowing us to examine and relate regularised boundary values of the Selberg integrals using the Drinfeld associator. Since the δ-function contains no poles that might lead to a suitable differential equation, further considerations are needed. We will comment on a possible workaround for this obstacle in section 5.3. Furthermore, it remains unclear which boundary values could yield promising results. In the original formulation of the Selberg integral, two natural choices were given by endpoints of the domain of integration, i.e. $x \to 0,1$. After compactification of the ζ_i integration from $\mathbb{R}$ to $[-\pi_{\max}^2, \zeta_{\tilde{\sigma}(i-1)}]$ in the MHV amplitude (cf. eq. (5.3)) , the δ-functions render the boundary values $\zeta_{\tilde{\sigma}(i-1)} \to -\pi_{\max}^2$ of the MHV gluon amplitude vanishing. In contrast to the discussion of the boundary values for the Selberg integrals, this limit seems impossible to regularise since the δ-function vanishes in an open neighbourhood of $-\pi_{\max}^2$. The alternative approach of approximating the δ-function by a normal distribution will be discussed further below.

Proceeding similarly to chapter 4, the computation of the derivative with respect to the point x_3 in eq. (4.5) depends on the vanishing of the Selberg seed as $x_i \to x_j$. This feature allows us to omit all the boundary terms that arise while integrating by parts. In section 5.1, we will propose a factor that generates δ-functions and vanishes as $\zeta_i \to \zeta_j$ for negative helicity particles associated to ζ_i and ζ_j. As defined in section 3.3.2, ζ_i denotes the coordinates $\lambda^2(z_i)/\lambda^1(z_i)$. The vertex operators representing negative helicity states come with the Grassmann fields $\psi^A(z_i)$, which naturally obey $\psi^A(\zeta_i)\psi^A(\zeta_j) \to 0$ as $\zeta_i \to \zeta_j$. Nevertheless, the factor we propose vanishes for two negative gluon vertices approaching each other *only*. This might be an indication that the recursive formalism we are examining is restricted to the insertion of negative helicity particles. If so, an amplitude with p positive and n negative helicities would correspond to the $(n-2)$-th step of the recursion with the MHV_{p+2} amplitude as its starting point.

Comparing Berkovits' formula in eq. (5.1) with the Selberg integrals, it is a priori unclear what role the relative distances $z_i - z_j$ from the latter play in the context of the δ-functions of the former. Clearly, the transition to relative coordinates is essential for following the lines of chapter 4. One viable coordinate transformation will be discussed in section 5.2, where we perform a substitution to simplicial coordinates. Other feasible choices of coordinates might include the dihedral coordinates introduced in [Bro09]. As we will show in section 5.2, the MHV gluon amplitude in simplicial coordinates contains factors of the form $\delta(x_i - x_{i+1})$. We note that, in this gluon sector, a basis consists of exactly one element, since the δ-functions force the integration variables x_i to condense on a single point. To draw the connection to the Selberg integral vectors (cf. eq. (4.8)), it might again be useful to approximate the δ-functions with smooth functions.

As mentioned above, the δ-functions appearing in a Selberg-like integral need to appear as derivatives of the Heaviside function. We argued that the δ-function itself does not allow for regularised boundary values at the endpoints of the domain of integration and that one might have to treat it as a limit of slowly decreasing functions. Instead of considering the δ-function as a limit of normal distributions, we may as well examine a smeared out Heaviside step function and its derivative. We may represent the latter as a limit of a complex integral

$$\theta(x) = \lim_{\epsilon\to 0^+} \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{e^{\mathrm{i}x\tau}}{\tau - \mathrm{i}\epsilon} \mathrm{d}\tau,$$

where, for $x > 0$, the contour is completed in the upper half of the complex plane containing the pole. For $x < 0$, the integrand diverges as $\tau \to +\mathrm{i}\infty$ and the integration must be extended by the infinite semi-circle in the lower half plane. Applying Cauchy's residue theorem for both cases yields

$$\theta(x) = \begin{cases} \lim_{\epsilon\to 0^+} \mathrm{Res}_{\tau=\mathrm{i}\epsilon}\left(\frac{e^{\mathrm{i}x\tau}}{\tau-\mathrm{i}\epsilon}\right) = \lim_{\epsilon\to 0^+} e^{-\epsilon x} = 1, & \text{for } x > 0, \\ 0, & \text{for } x < 0. \end{cases}$$

The ϵ-independent part of this representation is simply given by the Heaviside function. For a regularisation of the gluon amplitudes at certain boundary values, it would be desirable to have an x-dependent expression of both the δ- and the Heaviside function that allows for a soft approach of the integration variables. Another representation of the Heaviside function is given by the limit $\epsilon \to 0$ of

$$\theta_\epsilon(x) = \frac{1}{1 + e^{-x/\epsilon}}.$$

For each $\epsilon > 0$, we may define a sequence of functions approaching the δ-function as $\epsilon \to 0$

$$\delta_\epsilon(x) = \frac{e^{-x/\epsilon}}{\epsilon\left(1 + e^{-x/\epsilon}\right)^2}.$$

Neither the first nor the second representation yields a promising ϵ-expansion that allows us to smoothen the δ-functions. We conclude that further investigations are needed for regularising the boundary values of the amplitudes and that, for this work, the MHV amplitude cannot produce further insights by an approximation of the δ-function.

Conjecturing the existence of such a Selberg-like formulation of the gluon amplitudes was strongly inspired by the BCFW-relations used by Drummond and Henn in [DH09] for deriving a formula expressing all n-gluon tree-level amplitudes in terms of amplitudes with fewer external legs. Such a relation between different n-point scattering processes is what we are exploring in this work. In section 2.5 we reviewed the work of Drummond and Henn. A possible connection between the two approaches will be discussed in section 5.4.

A recursion formalism for the gluon amplitudes with the δ-functions taking on the role of the functions $1/x$ in the Selberg integrals requires additional insights into the structure of the integral over the algebraic curves in twistor space that are not discussed in this work. After a suitable transformation of the complex variables z_i in eq. (5.1), the amplitude can indeed be formulated as an iterated integral over a subset of $\mathbb{R}$. For the MHV amplitude, this substitution of variables is well understood, see eq. (3.10). Choosing the twistor space condition $\lambda^1(z) = 1$, it is given by a linear transformation. For degrees $d > 1$ on the other hand, higher order substitutions are necessary that complicate the algebraic structure of the integral immensely. This is the reason why we only considered the MHV amplitude in the following three subsections. We found that, at least for the MHV amplitude, the δ-functions alone do not carry the correct structure to apply the Selberg formalism from chapter 4 to the gluon amplitudes. As we will see in section 5.3, the necessary emergence of poles can be included by introducing a logarithm inside the Heaviside function. Since the δ-functions in the MHV amplitude force the iterated integral to condense on one single point, it remains unclear which boundary values a Drinfeld-like associator might connect. It is possible that the study of higher degrees of amplitudes yields a more suitable structure with more than one exceptional point that allows for a Selberg-like description. The identity for the δ-function

$$\delta(\pi_r^2 - \lambda^2(z)) = \sum_{i=1}^{d} \frac{\delta(z - z_i^r)}{\left|\lambda^{2\prime}(z_i^r)\right|}, \qquad \text{where } \pi_r^2 - \lambda^2(z_i^r) = 0,\ \forall 1 \le i \le d$$

allows us to replace each δ-function in the amplitude by a sum of d δ-functions. The d zeros of the r polynomials might form a set of interesting boundary values.

5.1 Iterating over the external momenta

The Selberg integrals we discussed in chapter 4 are iterated in the sense that the integration variables $z_i \in [0,1]$ are restricted to $z_i < z_{i-1}$ for $i \in \{3,\dots,L-1\}$. The gluon amplitudes in eq. (5.1) are given by integrals along algebraic curves of degree d. For the simple case of an MHV amplitude with $d = 1$, this means that all external momenta π_r lie on a straight line in twistor space along which we integrate. In this subsection, we try to identify the iterative structure inside such an MHV amplitude and take eq. (3.12) as the starting point. Let us restrict the complex spacetime to the real slice $\mathbb{R}^{2,2}$. As discussed in section 2.2, the spinors π are real in split signature (cf. eq. (2.7)). With the twistor space condition $\pi^1 = 1$ and the definition

$$A(r) = \begin{cases} 1 & \text{for } \epsilon_r = + \\ \prod_{b=1}^4 \psi^b(\zeta_r) & \text{for } \epsilon_r = - \end{cases}$$

we can express the amplitude as

$$\mathscr{A}^{\text{tree}}_{--+\dots+} = \int \mathrm{d}\zeta_1 \delta\left(\pi_1^2 - \zeta_1\right) \dots \int \mathrm{d}\zeta_n \delta\left(\pi_n^2 - \zeta_n\right) \langle 0|\, e^{q_0} \prod_{r=1}^{n} A(r)\, |0\rangle$$
$$\times \frac{f^{A_1\dots A_n}}{(\zeta_1 - \zeta_2)\dots(\zeta_n - \zeta_1)} \delta^4\left(\sum \pi_r \overline{\pi}_r\right),$$

where we took a step back by writing $(\zeta_1-\zeta_2)^4 = \langle 0|\, e^{q_0} \prod_{r=1}^n A(r)\,|0\rangle$ (cf. eq. (3.8)). Now that the spinors π_r are defined in a split signature spacetime and therefore real, the complex integrals over ζ_r can be seen as integrals along $\mathbb{R}$. Defining [1] the Heaviside function with $\theta(0)=0$, we are allowed to insert the exponential factor

$$\exp\left(\sum_{r=1}^{n} \theta\left(\zeta_r - \pi_r^2\right) A(r)\right) \tag{5.2}$$

into the inner product. This factor shall take on the role of the Selberg seed in eq. (4.3), whose derivatives generate the functions $1/x_{ij}$. Here, differentiation yields δ-functions, which are supposed to act as the analogue of $1/x_{ij}$ in the gluon amplitudes. When evaluating the integrals, this exponential factor is set to one by the δ-functions. Additionally, we want to decompose the integration area into simplices. First, let us limit the integration of each ζ_r to $\left[-\pi_{\max}^2, \pi_{\max}^2\right]$, where $\pi_{\max}^2 := \max\left\{\left|\pi_r^2\right| \middle| 1 \le r \le n\right\}$. The resulting n-dimensional hypercube can be decomposed into $n!$ simplices via the $n!$ permutations of the condition $\pi_{\max}^2 \ge \zeta_1 \ge \ldots \ge \zeta_n \ge -\pi_{\max}^2$. Thus,

$$\begin{aligned}\mathscr{A}^{\text{tree}}_{--+\ldots+} = \sum_{\sigma\in S_n} &\left[\int_{-\pi_{\max}^2}^{\pi_{\max}^2} \mathrm{d}\zeta_{\sigma(1)}\, \delta\left(\pi_{\sigma(1)}^2 - \zeta_{\sigma(1)}\right) \ldots \int_{-\pi_{\max}^2}^{\zeta_{\sigma(n-1)}} \mathrm{d}\zeta_{\sigma(n)}\, \delta\left(\pi_{\sigma(n)}^2 - \zeta_{\sigma(n)}\right)\right] \\ &\times \langle 0|\, e^{q_0} e^{\sum_{r=1}^n \theta(\zeta_r - \pi_r^2) A(r)} \prod_{r=1}^n A(r)\,|0\rangle \frac{f^{A_1\ldots A_n}}{(\zeta_1-\zeta_2)\ldots(\zeta_n-\zeta_1)} \delta^4\left(\sum \pi_r \overline{\pi}_r\right).\end{aligned}$$

Let us assume $i>j$ and that the permutation $\sigma_0 \in S_n$ obeys $\pi_{\sigma(i)}^2 > \pi_{\sigma(j)}^2$. Since $\zeta_{\sigma(i)} \le \zeta_{\sigma(j)}$, the function $\delta(\pi_{\sigma(j)}^2 - \zeta_{\sigma(j)})$ implies $\zeta_{\sigma(i)} \le \pi_{\sigma(j)}^2 < \pi_{\sigma(i)}^2$. This means that $\delta(\pi_{\sigma(i)}^2 - \zeta_{\sigma(i)})$ vanishes and that the permutation σ_0 does not contribute in the sum. Therefore, only one permutation $\widetilde{\sigma}$ with the property

$$\pi_{\widetilde{\sigma}(i)}^2 < \pi_{\widetilde{\sigma}(j)}^2 \qquad \text{for } i>j$$

yields a non-vanishing expression. Finally, we can write the amplitude as the iterated integral

$$\begin{aligned}\mathscr{A}^{\text{tree}}_{--+\ldots+} = &\int_{-\pi_{\max}^2}^{\pi_{\max}^2} \mathrm{d}\zeta_{\widetilde{\sigma}(1)}\, \delta\left(\pi_{\widetilde{\sigma}(1)}^2 - \zeta_{\widetilde{\sigma}(1)}\right) \ldots \int_{-\pi_{\max}^2}^{\zeta_{\widetilde{\sigma}(n-1)}} \mathrm{d}\zeta_{\widetilde{\sigma}(n)}\, \delta\left(\pi_{\widetilde{\sigma}(n)}^2 - \zeta_{\widetilde{\sigma}(n)}\right) \\ &\times \langle 0|\, e^{q_0} e^{\sum_{r=1}^n \theta(\zeta_r - \pi_r^2) A(r)} \prod_{r=1}^n A(r)\,|0\rangle \frac{f^{A_1\ldots A_n}}{(\zeta_1-\zeta_2)\ldots(\zeta_n-\zeta_1)} \delta^4\left(\sum \pi_r \overline{\pi}_r\right).\end{aligned} \tag{5.3}$$

Although we wrote the MHV amplitude as an iterated integral, the algebraic structure still differs strongly from that of a Selberg integral. The most obvious discrepancy is the disconnection of the integration variables ζ_i, which are coupled to their respective external momenta π_i instead of to themselves. The substitution in the following subsection will yield variables x_i that are indeed connected to themselves, meaning that the δ-functions restrict the domain of integration to $x_i = x_j\ \forall i,j$.

[1] Any other convention is also possible. For $r \in \{1,2\}$, the operator e^{q_0} ensures that the expansion of $\exp\left(\theta(\pi_r^2 - \zeta_r) A(r)\right)$ is terminated after the 0th order. For the remaining values of r we have to take into account the $n-2$ factors of $e^{\theta(0)}$.

5.2 Transformation to simplicial coordinates

In order to draw the connection to the Selberg integral formalism introduced in the previous section, we need to write the expressions $\zeta_i - \pi_i^2 = 0$ as differences of the coordinates, such as $x_i - x_j = 0$. For this purpose, we define

$$z_i := \prod_{j=i}^{n} \zeta_j, \qquad \alpha_i := \prod_{j=i} \pi_j^2. \tag{5.4}$$

The Jacobian of this substitution inside eq. (3.12) reads

$$\left| \left(\frac{\mathrm{d}z_h}{\mathrm{d}\zeta_k} \right)_{h,k} \right| = \left\| \begin{pmatrix} z_2 & \star & \star & \star \\ 0 & z_3 & \star & \star \\ \vdots & \ddots & \ddots & \star \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\| = \prod_{i=2}^{n} z_i.$$

It follows that the differential from eq. (3.12) can be written as

$$\begin{aligned} \prod_{i=1}^{n} \int \mathrm{d}\zeta_i \delta(\zeta_i - \pi_i^2) &= \prod_{i=1}^{n} \int \frac{\mathrm{d}z_i}{z_{i+1}} \delta\left(\frac{z_i}{z_{i+1}} - \frac{\alpha_i}{\alpha_{i+1}} \right) \\ &= \prod_{i=1}^{n} \int \frac{\mathrm{d}z_i}{\alpha_i} \delta\left(\frac{z_i}{\alpha_i} - \frac{z_{i+1}}{\alpha_{i+1}} \right), \end{aligned}$$

where we pulled out a factor $\frac{\alpha_i}{z_{i+1}}$ from the δ-function and denoted $z_{n+1} \equiv 1$. Denoting $x_i \equiv \frac{z_i}{\alpha_i}$, we reach the form

$$\prod_{i=1}^{n} \int \mathrm{d}x_i \delta(x_i - x_{i+1}).$$

We note that after a linear transformation of the ζ_is such that $\zeta_i \in [0,1]$, $\forall i$, the simplicial coordinates satisfy $0 \leq x_{i+1} \leq x_i \leq 1$. This implies that each variable x_i is integrated from 0 to x_{i-1} and that the iterative character of the integral is implemented automatically. Therefore, the decomposition into simplices from the previous subsection is redundant.

This substitution brings us one step closer to a Selberg integral as in eq. (4.2). The integration variables appear pairwise inside the δ-function instead of together with their external momenta π_r^2. In the next subsection, we will account for the deficiency of poles that are crucial for the introduction of the Drinfeld associator (cf. section 4.3) and arise after the differentiation of the Selberg seed.

5.3 Logarithmic scaling inside the Heaviside function

As discussed before, another missing piece for formulating the gluon amplitudes as Selberg integrals is the absence of poles in the differentiation of the exponential factor in eq. (5.2) that is supposed to take on the role of the Selberg seed in chapter 4. Since the δ-function

$$\lim_{\epsilon\to 0}(\epsilon\sqrt{\pi})^{-1}e^{-x^2/\epsilon^2}$$

includes no poles, the establishment of a KZ-equation seems troublesome. One way to avoid this would be the inclusion of a logarithm inside the Heaviside function, where the inner derivative of the logarithm generates a pole. Omitting the factor $A(r)$, the exponential factor we introduced in eq. (5.2) can be rewritten using the simplicial coordinates defined in eq. (5.4) as

$$\begin{aligned}\exp\left(\sum_{r=1}^{n}\theta\left(\zeta_r-\pi_r^2\right)\right)&=\exp\left(\sum_{r=1}^{n}\theta\left(\frac{z_r}{z_{r+1}}-\frac{\alpha_r}{\alpha_{r+1}}\right)\right)\\&=\exp\left(\sum_{r=1}^{n}\theta\left(\frac{z_r}{\alpha_r}-\frac{z_{r+1}}{\alpha_{r+1}}\right)\right)\\&=\exp\left(\sum_{r=1}^{n}\theta\left(x_r-x_{r+1}\right)\right),\end{aligned}$$

where we pulled the (positive) factors $\frac{z_{r+1}}{\alpha_r}$ out of the Heaviside functions. In analogy to the Selberg seed in eq. (4.3) where the sum ranges over all $x_i < x_j$ instead of the adjacent insertions, we extend the exponential factor accordingly, yielding

$$\exp\left(\sum_{i=1}^{n}\sum_{j=i+1}^{n}\theta\left(x_i-x_j\right)\right).$$

For a derivative to generate a pole, we let the difference $x_i - x_j$ inside the Heaviside function scale logarithmically. Due to the identity

$$\delta(g(x))=\sum_i\frac{\delta(x-x_i)}{\left|g'(x_i)\right|},\qquad g(x_i)=0,$$

the expression inside the Heaviside function that generates the original function $\delta(x_i-x_j)$ after differentiation must have a zero at $x_i = x_j$. Including a factor $\langle ij\rangle$ for reasons that will become apparent shortly, the factor should be of the form

$$\exp\left(\sum_{i=1}^{n}\sum_{j=i+1}^{n}\langle ij\rangle\theta\left(\log(x_i-x_j+1)\right)\right).$$

The derivative of this exponential factor with respect to one of the insertion points reads

$$\begin{aligned}&\frac{\partial}{\partial x_m}\exp\left(\sum_{i=1}^{n}\sum_{j=i+1}^{n}\langle ij\rangle\theta\left(\log(x_i-x_j+1)\right)\right)=\\&=\left(\sum_{j=m+1}^{n}\frac{\langle mj\rangle}{x_m-x_j+1}\delta(x_j-x_m)-\sum_{i=1}^{m-1}\frac{\langle im\rangle}{x_i-x_m+1}\delta(x_m-x_i)\right)\\&\quad\times\exp\left(\sum_{i=1}^{n}\sum_{j=i+1}^{n}\langle ij\rangle\theta\left(\log(x_i-x_j+1)\right)\right)\\&=\left(\sum_{l\neq m}\frac{\langle lm\rangle}{x_l-x_m+1}\delta(x_l-x_m)\right)\exp\left(\sum_{i=1}^{n}\sum_{j=i+1}^{n}\langle ij\rangle\theta\left(\log(x_i-x_j+1)\right)\right),\end{aligned}$$

where we used the antisymmetry of the spinor bracket $\langle ij\rangle = -\langle ji\rangle$ and $\frac{\delta(x_i-x_j)}{x_i-x_j+1} = \frac{\delta(x_j-x_i)}{x_j-x_i+1}$. Similar to the differentiation of the Selberg seed in eq. (4.6), here, the derivative generates a simple pole together with the desired δ-function. As we discussed before, the MHV amplitude condenses on a single point and no further distinguished points appear, whose boundary values might be connected via a Drinfeld associator. Nevertheless, we established a factor that has the necessary properties for taking on the role of the Selberg seed in higher order amplitudes.

5.4 Identification of iterative methods in gluon amplitudes

As the discussion in the previous subsections illustrate, the δ-function might not carry the sufficient structure needed for a Selberg-like formulation of the gluon amplitudes. Consequently, in the rest of this work, we investigate an alternative approach. The BCFW relations for gluon scattering in section 2.5 express the n-particle amplitudes as functions of the amplitudes with fewer external legs. This subsection is concerned with the connection between said discussion and the recursion formalism for Selberg integrals in chapter 4. To enhance the readability of the following, we write for the r.h.s. of eq. (2.16)

$$\mathscr{A}_i \odot \mathscr{A}_j := \int \frac{\mathrm{d}^4 P}{P^2} \int \mathrm{d}^4 \eta_P \mathscr{A}_i(z) \mathscr{A}_j(z). \tag{5.5}$$

Here, we included an integration over the momentum P to combine the two momentum conserving δ-functions of $\mathscr{A}_i(z)$ and $\mathscr{A}_j(z)$, which was done implicitly in eq. (2.16). Since the shift variable z is determined by the poles of the propagator (cf. eq. (2.14)), the product is indeed well-defined. For example, the NMHV amplitude in eq. (2.17) is written as

$$\mathscr{A}_n^{\text{NMHV}} = \mathscr{A}_3^{\overline{\text{MHV}}} \odot \mathscr{A}_{n-1}^{\text{NMHV}} + \sum_{i=4}^{n-1} \mathscr{A}_i^{\text{MHV}} \odot \mathscr{A}_{n-i+2}^{\text{MHV}}$$

We investigate amplitudes that arise after appending a positive or negative gluon to the amplitude $--+\ldots+$, i.e. the ones of the form $--+\ldots+\pm$. Thus, the foundation of the recursion consists of the well-known MHV amplitudes accompanied by $\overline{\text{MHV}_3}$ $(-++)$. For five external legs, we have the relations

$$\begin{aligned}
\mathscr{A}_5^{\text{MHV}} &= \frac{\langle 41\rangle}{\langle 51\rangle\langle 45\rangle} \mathscr{A}_4^{\text{MHV}}, \\
\mathscr{A}_5^{\text{NMHV}} &= \mathscr{A}_3^{\overline{\text{MHV}}} \odot \underbrace{\mathscr{A}_4^{\text{NMHV}}}_{=0} + \mathscr{A}_4^{\text{MHV}} \odot \mathscr{A}_3^{\text{MHV}},
\end{aligned}$$

where the amplitude $\mathscr{A}_4^{\text{NMHV}}$ with helicities $---+$ vanishes (cf. eq. (2.11)). The six-point amplitudes are given by (cf. eq. (2.18))

$$\begin{aligned}
\mathscr{A}_6^{\text{MHV}} &= \frac{\langle 51\rangle}{\langle 61\rangle\langle 56\rangle}\mathscr{A}_5^{\text{MHV}}, \\
\mathscr{A}_6^{\text{NMHV}} &= \mathscr{A}_3^{\overline{\text{MHV}}} \odot \mathscr{A}_5^{\text{NMHV}} + \mathscr{A}_4^{\text{MHV}} \odot \mathscr{A}_4^{\text{MHV}} + \mathscr{A}_5^{\text{MHV}} \odot \mathscr{A}_3^{\text{MHV}}, \\
\mathscr{A}_6^{\text{NNMHV}} &= \mathscr{A}_3^{\overline{\text{MHV}}} \odot \underbrace{\mathscr{A}_5^{\text{NNMHV}}}_{=0} + \mathscr{A}_5^{\text{NMHV}} \odot \mathscr{A}_3^{\text{MHV}},
\end{aligned}$$

where again $\mathscr{A}_5^{\text{NNMHV}} = 0$ due to eq. (2.11).

The Selberg formalism including the fulfilling of the KZ equation in eq. (4.9) and the Drinfeld associator connecting the two boundary values in eq. (4.11) describes a vector of Selberg integrals (cf. eq. (4.8)). Analogously, we establish a vector of gluon amplitudes and search for a relation between different boundary values of the shifted amplitudes $\mathscr{A}(z)$ and a differential equation obeyed by this vector. We define

$$\mathbf{A}_n = \begin{pmatrix} \mathscr{A}_3^{\overline{\text{MHV}}} \\ \mathscr{A}_3^{\text{MHV}} \\ \mathscr{A}_4^{\text{MHV}} \\ \mathscr{A}_5^{\text{MHV}} \\ \mathscr{A}_5^{\text{NMHV}} \\ \vdots \\ \mathscr{A}_n^{\text{N}^{n-4}\text{MHV}} \end{pmatrix}.$$

The relations for the five- and six-point amplitudes may then be written as

$$\mathbf{A}_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{\langle 41\rangle\delta(p_5-P)}{\langle 51\rangle\langle 45\rangle} & 0 \\ 0 & \mathscr{A}_4^{\text{MHV}} & 0 \end{pmatrix} \odot \mathbf{A}_4,$$

$$\mathbf{A}_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{\langle 51\rangle\delta(p_6-P)}{\langle 61\rangle\langle 56\rangle} & 0 \\ 0 & \mathscr{A}_5^{\text{MHV}} & \mathscr{A}_4^{\text{MHV}} & 0 & \mathscr{A}_3^{\overline{\text{MHV}}} \\ 0 & 0 & 0 & 0 & \mathscr{A}_3^{\text{MHV}} \end{pmatrix} \odot \mathbf{A}_5.$$

The appearance of $\delta(p_n - P)$ is due to the integration over P in eq. (5.5) and replaces P with p_n in the momentum conserving δ-function of the amplitude inside the vector. We note that the matrix relating the n-point and the $n-1$-point amplitude vectors can always be written in terms of $\frac{\langle n-1,1\rangle}{\langle n1\rangle\langle n-1,n\rangle}$ and $(n-2)$-point amplitudes. In this convention, the vector relations from above read

$$\mathbf{A}_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{\langle 41\rangle\delta(p_5-P)}{\langle 51\rangle\langle 45\rangle} & 0 \\ 0 & 0 & \mathscr{A}_3^{\text{MHV}} \end{pmatrix} \odot \mathbf{A}_4,$$

$$\mathbf{A}_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{\langle 51\rangle\delta(p_6-P)}{\langle 61\rangle\langle 56\rangle} & 0 \\ 0 & 0 & \mathscr{A}_4^{\text{MHV}} & \mathscr{A}_3^{\text{MHV}} & \mathscr{A}_3^{\overline{\text{MHV}}} \\ 0 & 0 & 0 & 0 & \mathscr{A}_3^{\text{MHV}} \end{pmatrix} \odot \mathbf{A}_5.$$

The BCFW relations express an amplitude as products of i-point and j-point amplitudes with $i + j = n + 2$ with $i, j < n$. Therefore, all $n-1$-point amplitudes may be seen as a contribution of the vector and do not need to appear inside the matrix.

The amplitudes on the r.h.s. of these relations are shifted with $z \neq 0$, while the l.h.s. denotes unshifted amplitudes with $z = 0$. Furthermore, each $(n-1)$-point vector on the r.h.s. can be replaced by the corresponding n-point vector after expanding the matrix by $n-3$ columns of zeros. We will call this extended matrix $\Phi_{n-1,n}$. The resulting equations relating different limits of the shifted amplitude vectors $\mathbf{A}_n(z)$

$$\mathbf{A}_n(z=0) = \Phi_{n-1,n}\mathbf{A}_n\left(\{z_i\}_i\right)$$

strongly resembles the Drinfeld associator equation in eq. (4.11). It is important to note that the amplitudes $A(z)$ on the r.h.s. do not possess the same shift parameter z, which is determined by the on-shell condition of each specific amplitude. Nevertheless, we can reformulate the shifting of each amplitude with a scaled parameter z_0 such that $\mathscr{A}(z_0 = 1)$ is on-shell and $\mathscr{A}(z_0 = 0) = \mathscr{A}(z = 0)$. This is to say that the vector $\mathbf{A}_n$ on the r.h.s. is shifted with $z_0 = 1$ while the l.h.s. is unshifted with $z_0 = 0$. The matrix connecting these two boundary values of $\mathbf{A}_n$ may be identified with the Drinfeld associator for the Selberg integrals. It remains unclear which KZ-like differential equation for the gluon amplitude vector might induce this connector.

5.5 Conclusion

The aim of this work was to find a connection between the $N = 4$ SYM gluon amplitudes and the Selberg integral formalism that was reviewed in chapter 4. The algebraic structure of the amplitudes as curves of degree $d = q - 1$, where q denotes the number of negative helicity particles, indicates that they may be formulated as iterated integrals along that curve. We suspected that the δ-functions restricting the integration over the complex plane of the vertex operators to this (complex) one dimensional surface might take on the role of the functions $1/x_{ij}$ in the Selberg integrals (cf. eq. (4.2)). Similar to the Selberg seed from eq. (4.3) that generates the function $1/x_{ij}$ after differentiation, an analogous factor in the gluon amplitudes must therefore include a Heaviside function as the distributional antiderivative of the δ-function.

We focused on the MHV amplitudes where a linear transformation of the integration variables suffices to bring the integral in an iterated form. The N^kMHV amplitudes with $k \geq 1$ require polynomial transformations of higher order that hinder their formulation as iterated integrals. Following the derivation of the Drinfeld associator that connects different regularised limits of the Selberg integrals, it remains unclear which boundary values of an extra point x_3 in the MHV amplitude might reveal a recursive structure of the gluon amplitudes. As we have shown in sections 5.1 and 5.2, the integrals condense on a single point which acts as the only special boundary value of the amplitude. The non-linearity of the substitutions in N^kMHV amplitudes might yield more than one such point whose boundary values are connected via a matrix similar to the Drinfeld associator. A suitable vector consisting of gluon amplitudes with boundary values connected in that way has to be a solution to the KZ equation (cf. eq. (4.9)) and the differentiation of the amplitude thus needs to generate a simple pole. We found that, by scaling the argument of the Heaviside function logarithmically, differentiation yields a δ-function combined with a simple pole. By studying the MHV amplitude, we established restrictions on a Selberg seed in SYM-theory that will be of use for the investigation of higher order amplitudes which may allow for a suitable structure with more than one distinguished boundary value.

Due to the structure of the MHV amplitude and the algebraic intricacy of higher order amplitudes, we took another approach by investigating the gluon amplitude recursion relations in [DH09] with the objective to identify them with a Selberg-like structure. We found that the BCFW relations give rise to a vector of gluon amplitudes whose boundary values with respect to the BCFW shift parameter z are connected via a matrix consisting of lower order gluon amplitudes. This connection between different boundary values of said vector is another indication that there might exist a description of the gluon amplitudes similar to the Selberg integrals and the Drinfeld associator and motivates the search for a suitable KZ-like differential equation satisfied by the gluon amplitude vector in the future.

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